Double-Categorical Compositional Rewriting Theory

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JWW: R. Harmer (ENS Lyon)

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J. Kock (UAB Barcelona)

References :

Fundamentals of Compositional Rewriting Theory*

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Tracelet Hopf Algebras and Decomposition spaces (Extended Abstract)

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Convolution Products on Double Categories and Categorification of Rule Algebras

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On Stochastic Rewriting and Combinatorics via Rule-Algebraic Methods*

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1 MOTIVATION: COMPOSITIONALITY & COMBINATORICS in REWRITING SYSTEMS

$$\begin{tikzcd} O & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y \arrow[from=1-1, "n"'] \arrow[ur, phantom, "\Downarrow\alpha"] & X \arrow[l, "r_{\alpha}"] \end{tikzcd}$$

a rewriting operation ("DIRECT DERIVATION")

COMBINATORICS: "# of ways to rewrite X to Y?"

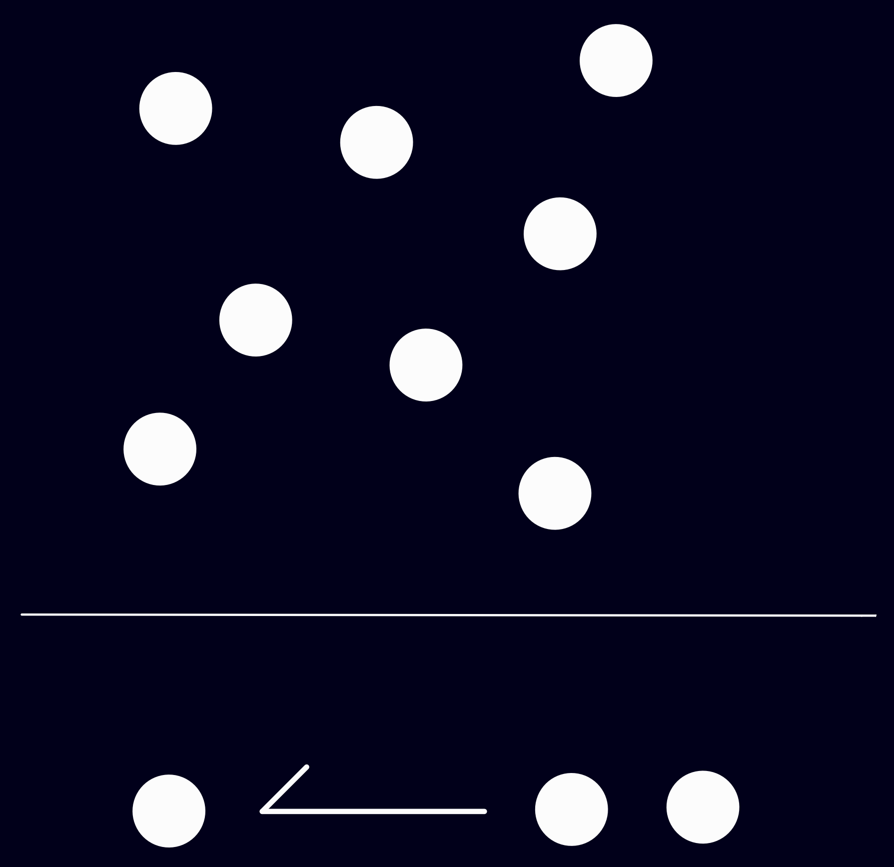
TRANSITION SYSTEMS

Markov chains, automata, MDPs, ...

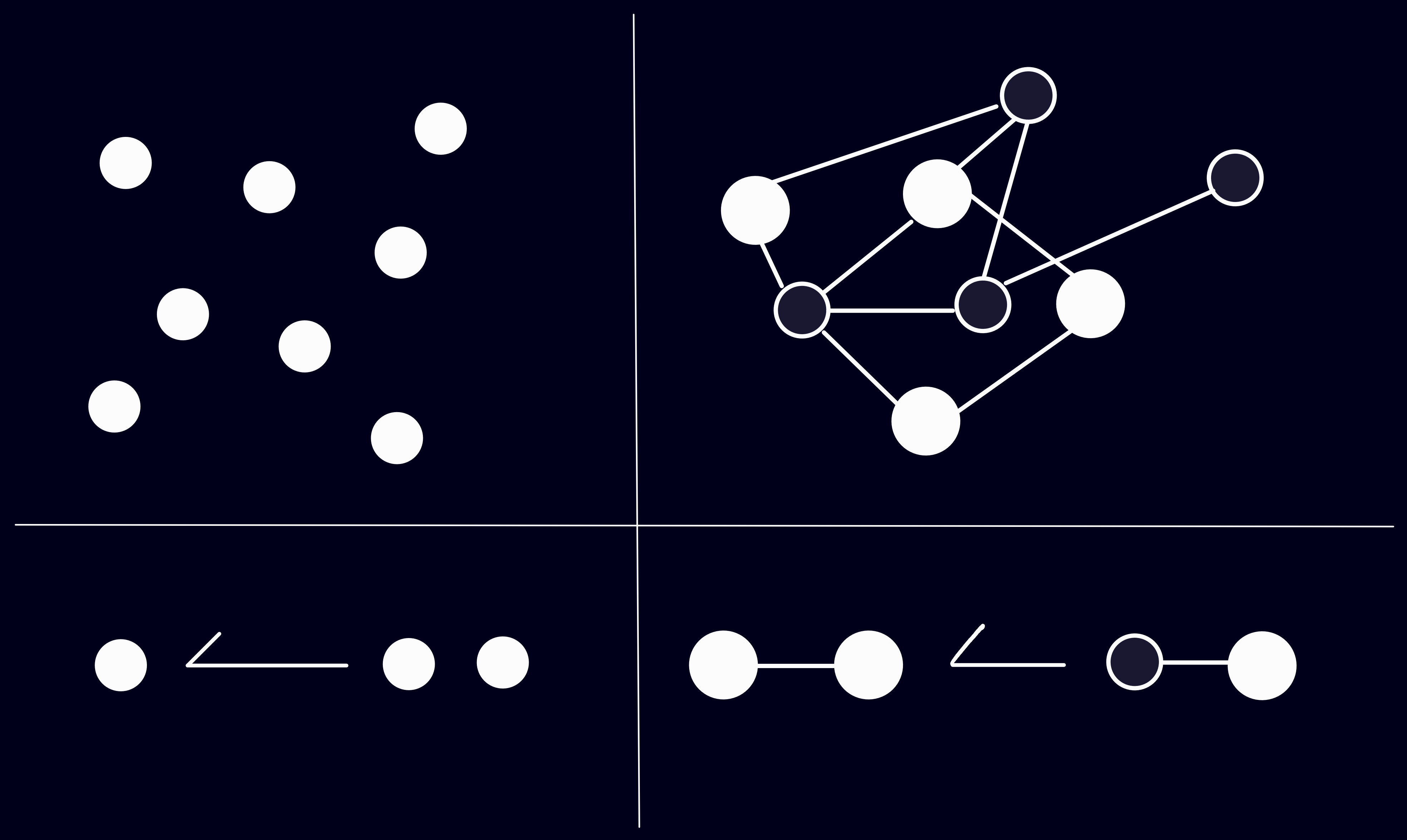
PROOF SYSTEMS

(based on diagrammatic "proof rules" ...)

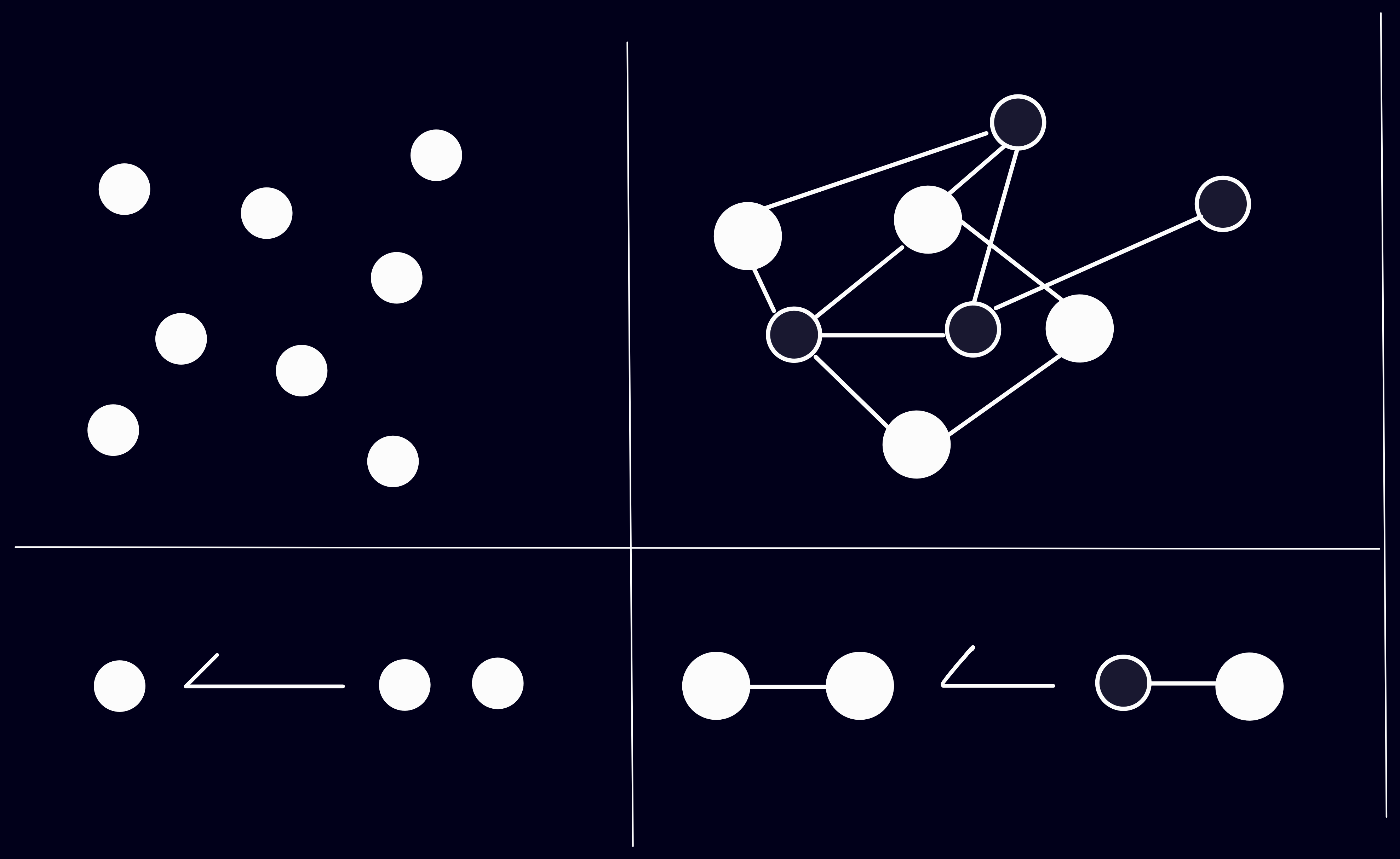
1 Motivation

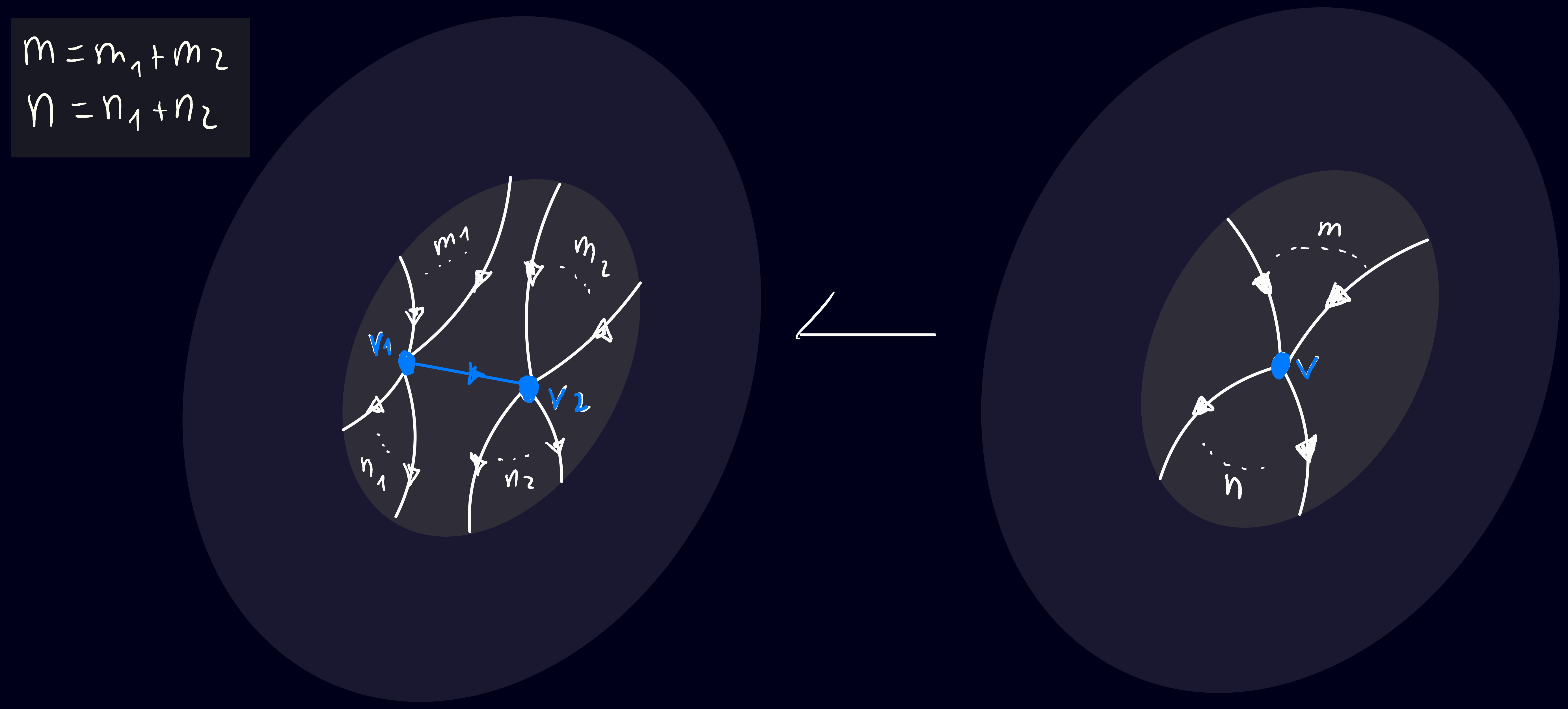


1 MOTIVATION

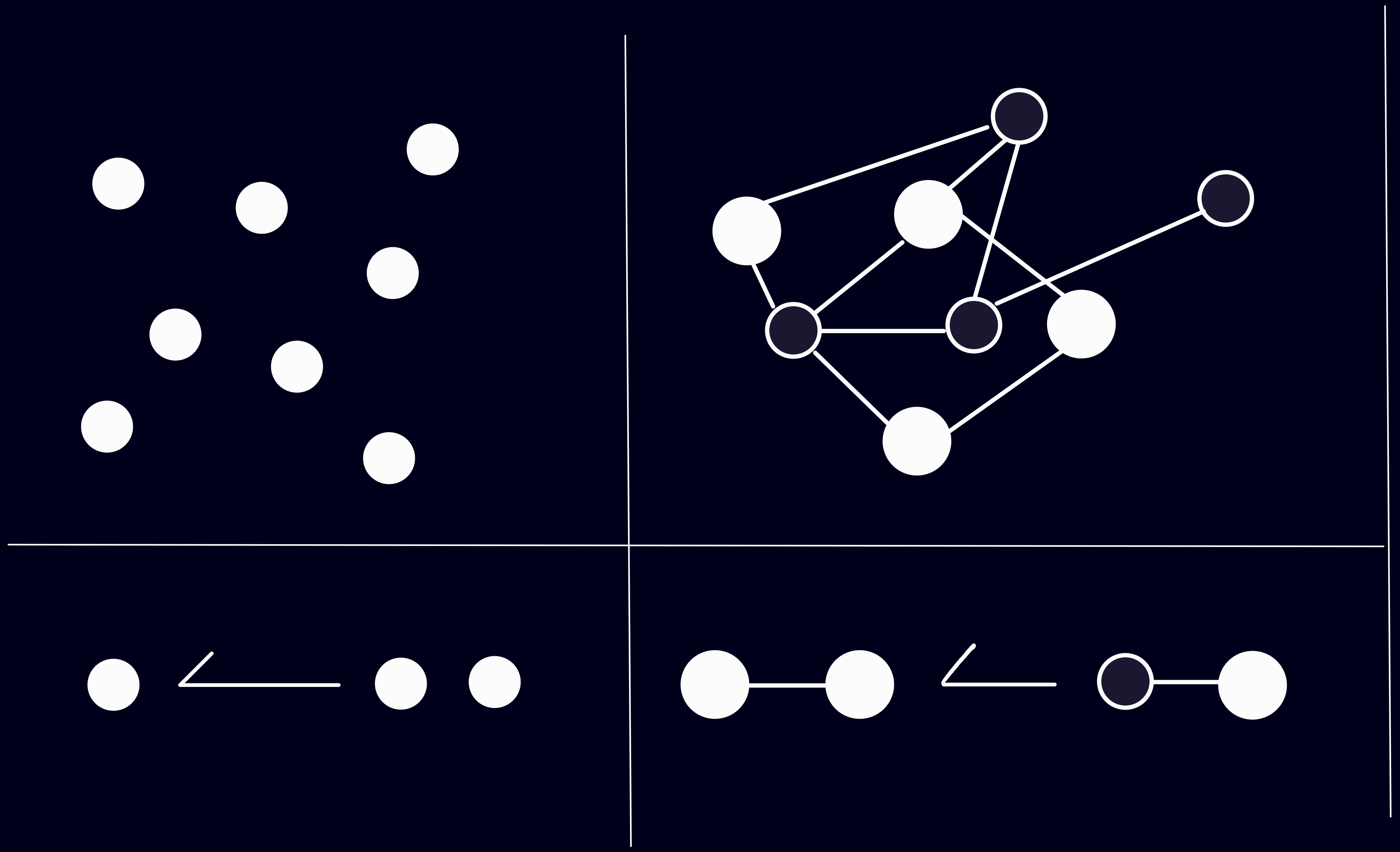


1 MOTIVATION

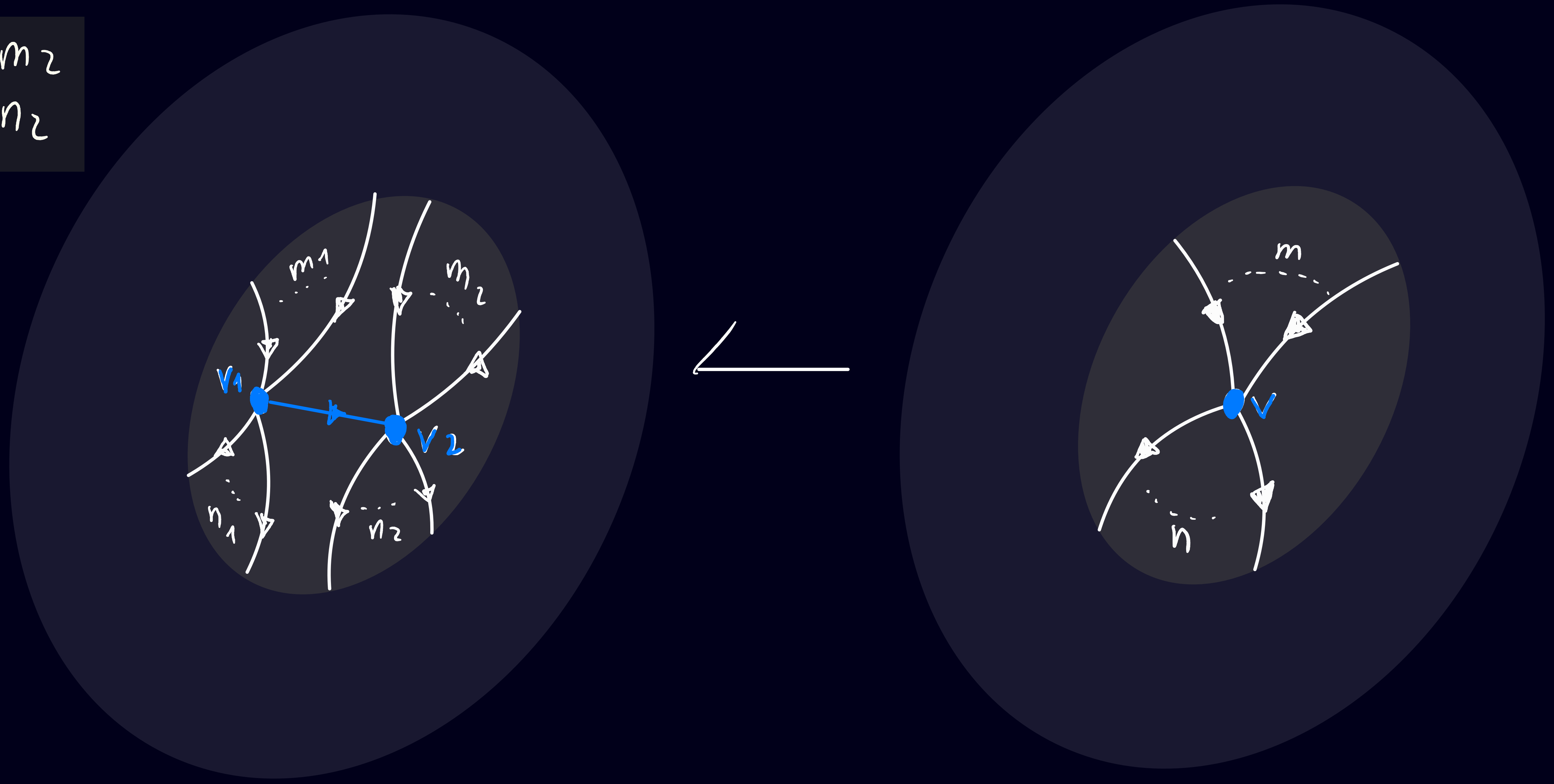




1 Motivation



$m = m_1 + m_2$
$n = n_1 + n_2$



↳ All formalizable in Double-Pushout (DPO) semantics:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
r_\alpha(X) & X \arrow[l]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — Pushout

CONCEPTUAL OBSTACLE: ESSENTIAL UNIQUENESS OF UNIVERSAL CONSTRUCTIONS

RECAP:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
r_\alpha(X) & X \arrow[l]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — PUSHOUT

DEFINITION:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\text{ is a PO } :\Leftrightarrow \forall
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D & \\
& & X
\end{tikzcd}
:
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}
$$

$$
\hookrightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\wedge
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r] & D'
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!", "\cong"'] & \\
& & D'
\end{tikzcd}
$$

EXAMPLE: (in FinSet)

$$
\begin{tikzcd}
I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r, hook] & \text{"}B \cup_I C\text{"}
\end{tikzcd}
$$

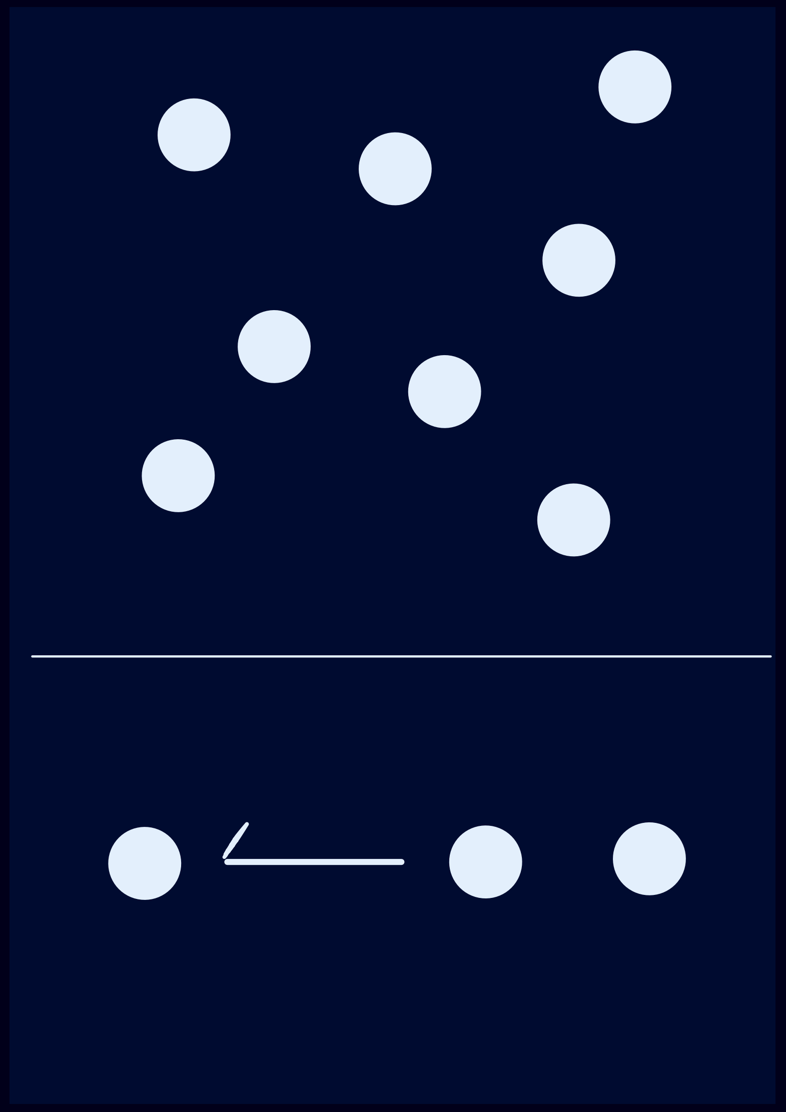
Conceptual Obstacle: Essential Uniqueness of Universal Constructions

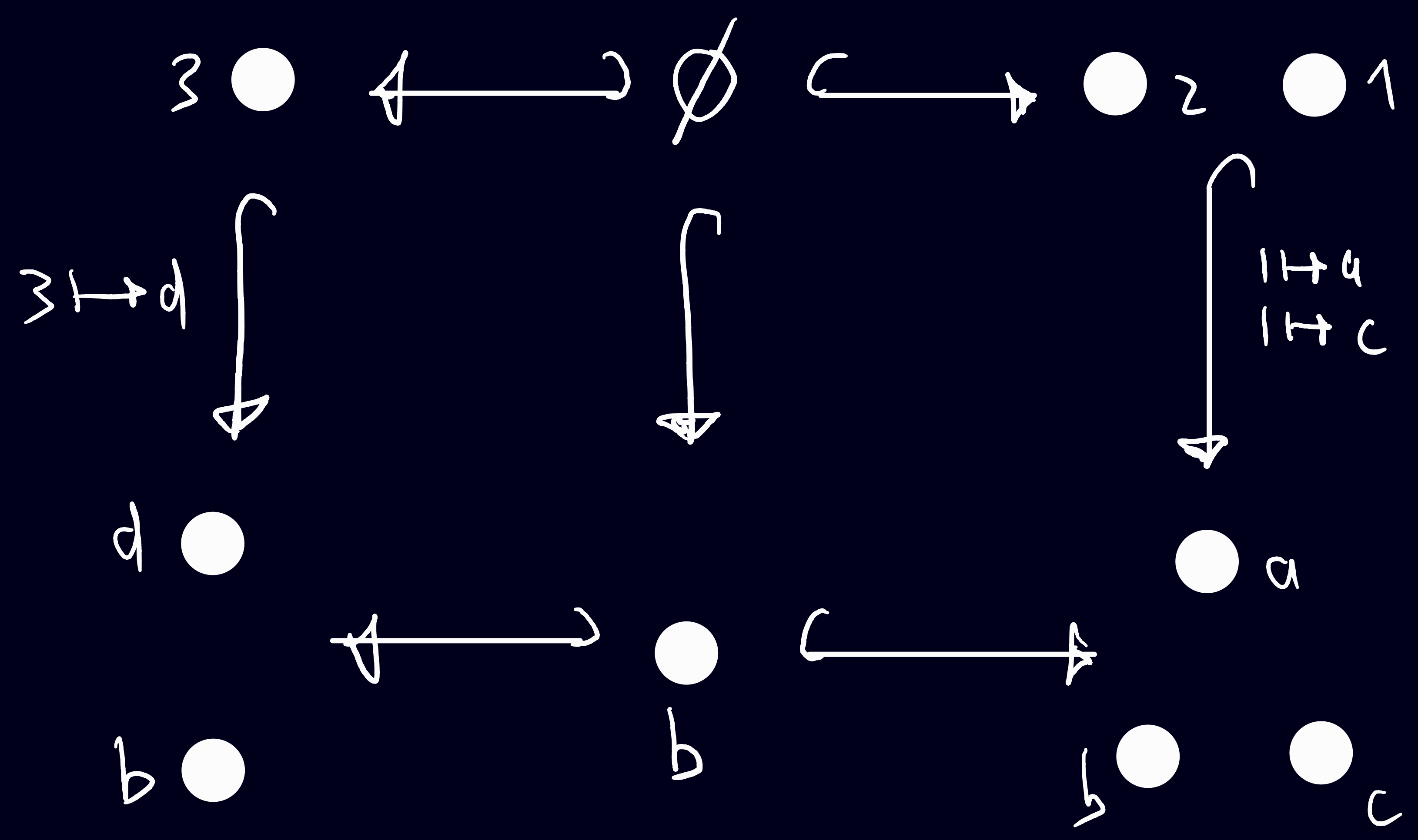
Recap:

$$\begin{tikzcd} O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd} \quad := \quad \begin{tikzcd} O \arrow[d, "n"'] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dl, phantom, "PO"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\ r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X \end{tikzcd}$$

PO — Pushout

Example:

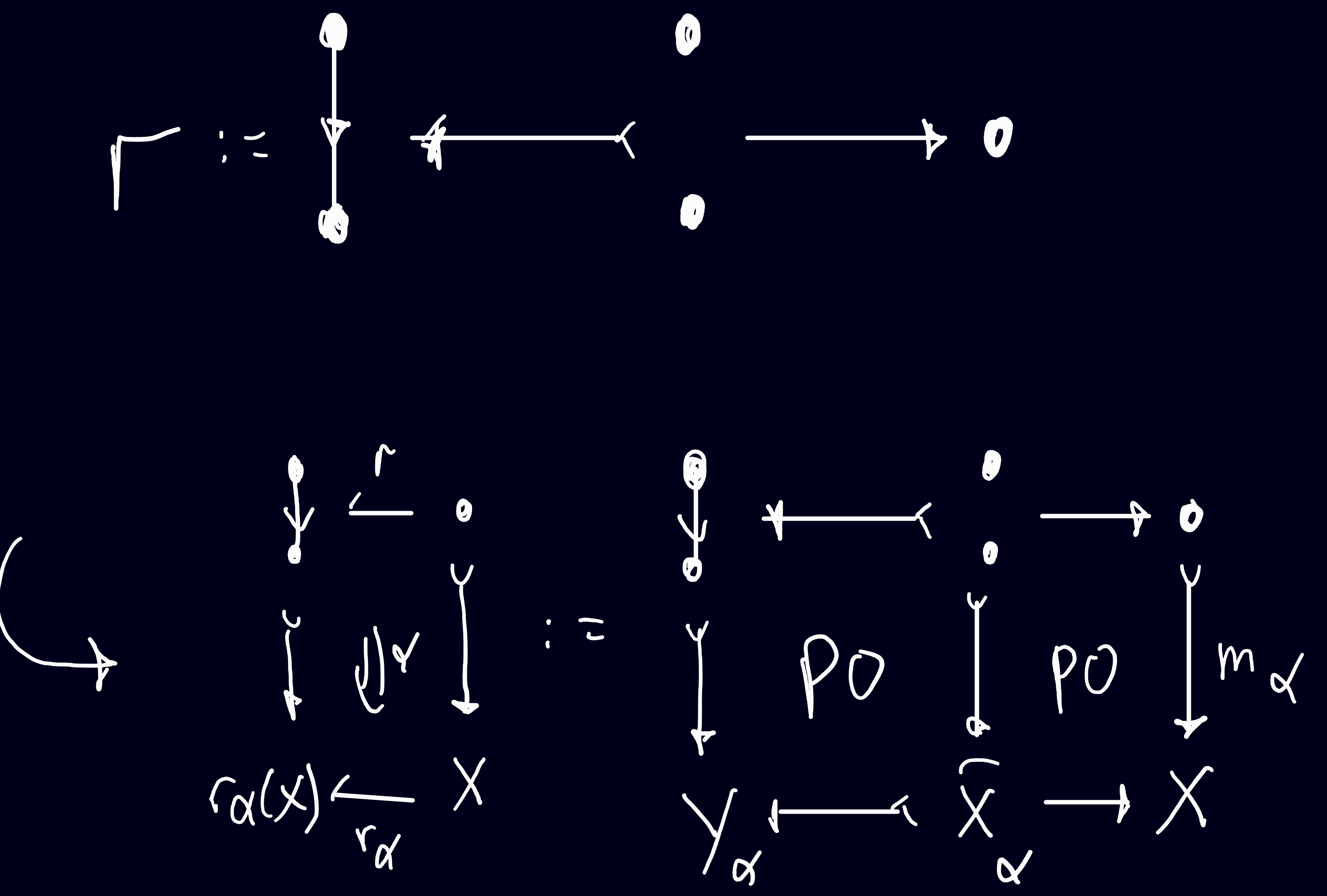




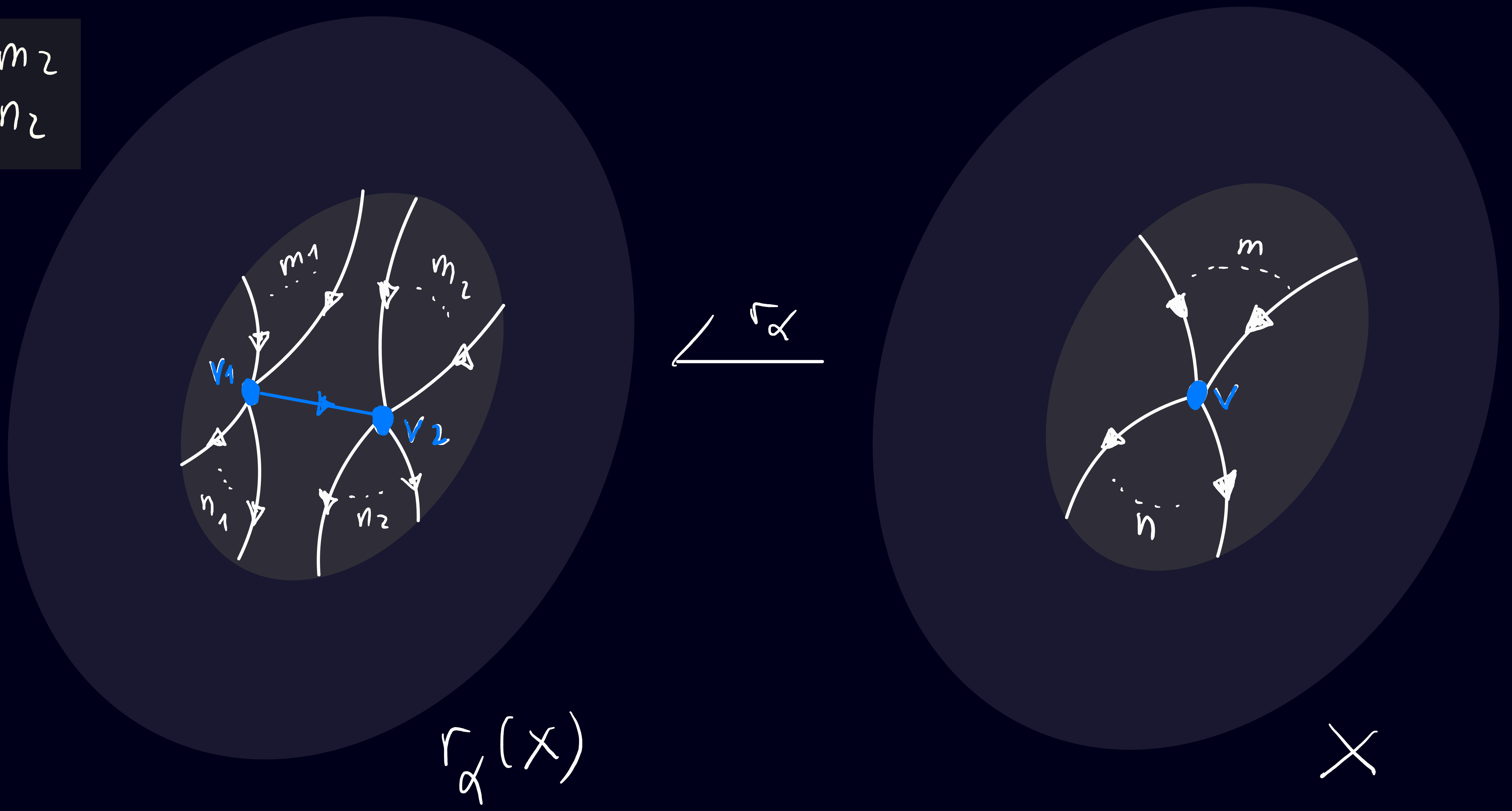
In FinSet:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "PO"] & B \arrow[d, hook] \\ C \arrow[r, hook] & \text{"}B \cup_I C\text{"} \end{tikzcd}$$

3 "Non-linear" rules in Kontsevich's graph cohomology calculus



$m = m_1 + m_2$
$n = n_1 + n_2$

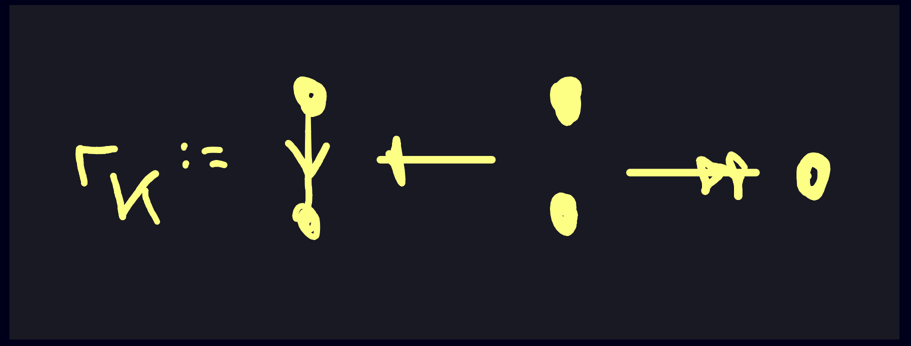


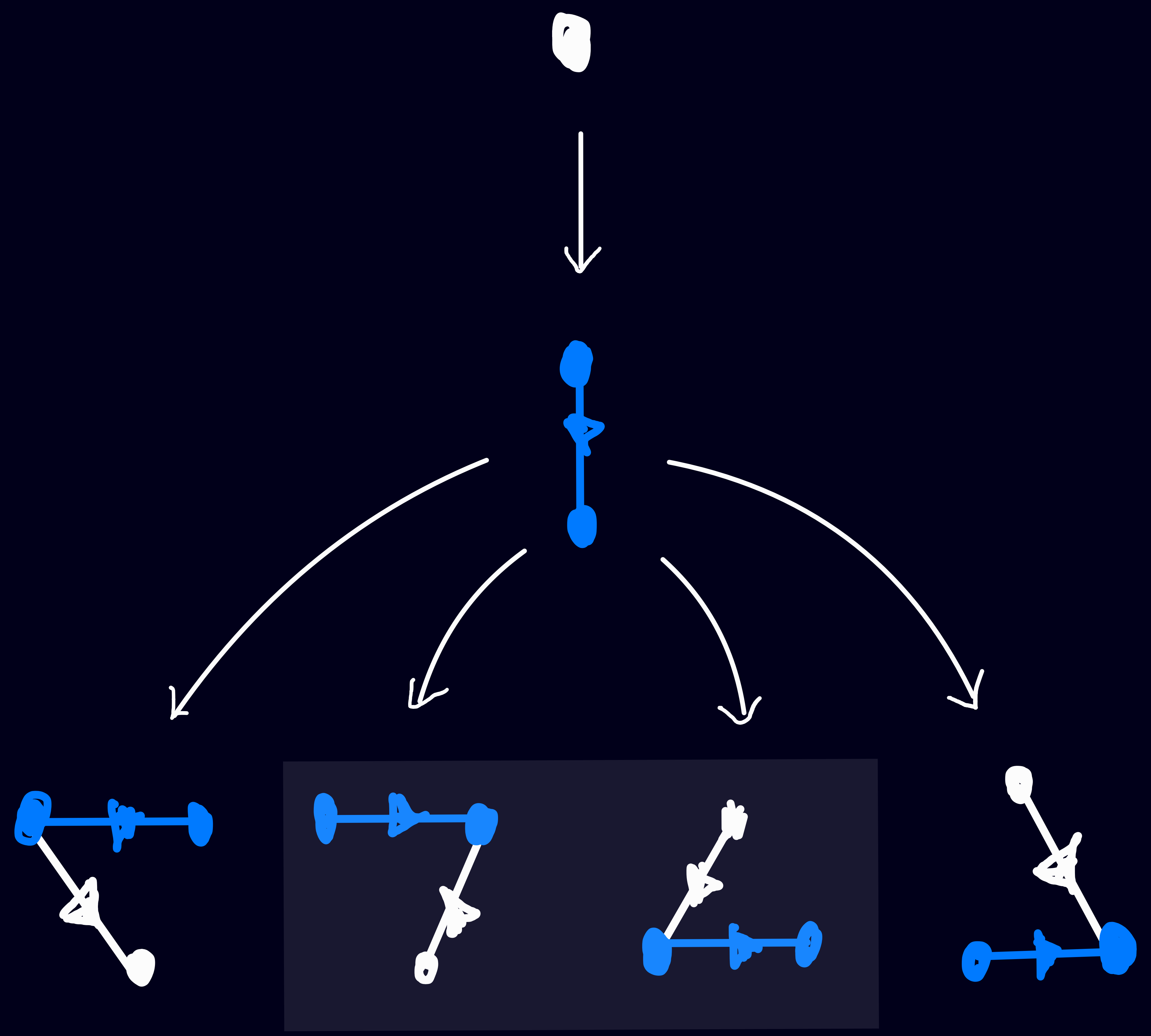
- Observation: up to isomorphisms, each rewrite is determined by:

  (i) Choice of vertex in X $(\bullet \xrightarrow{m_\alpha} X)$

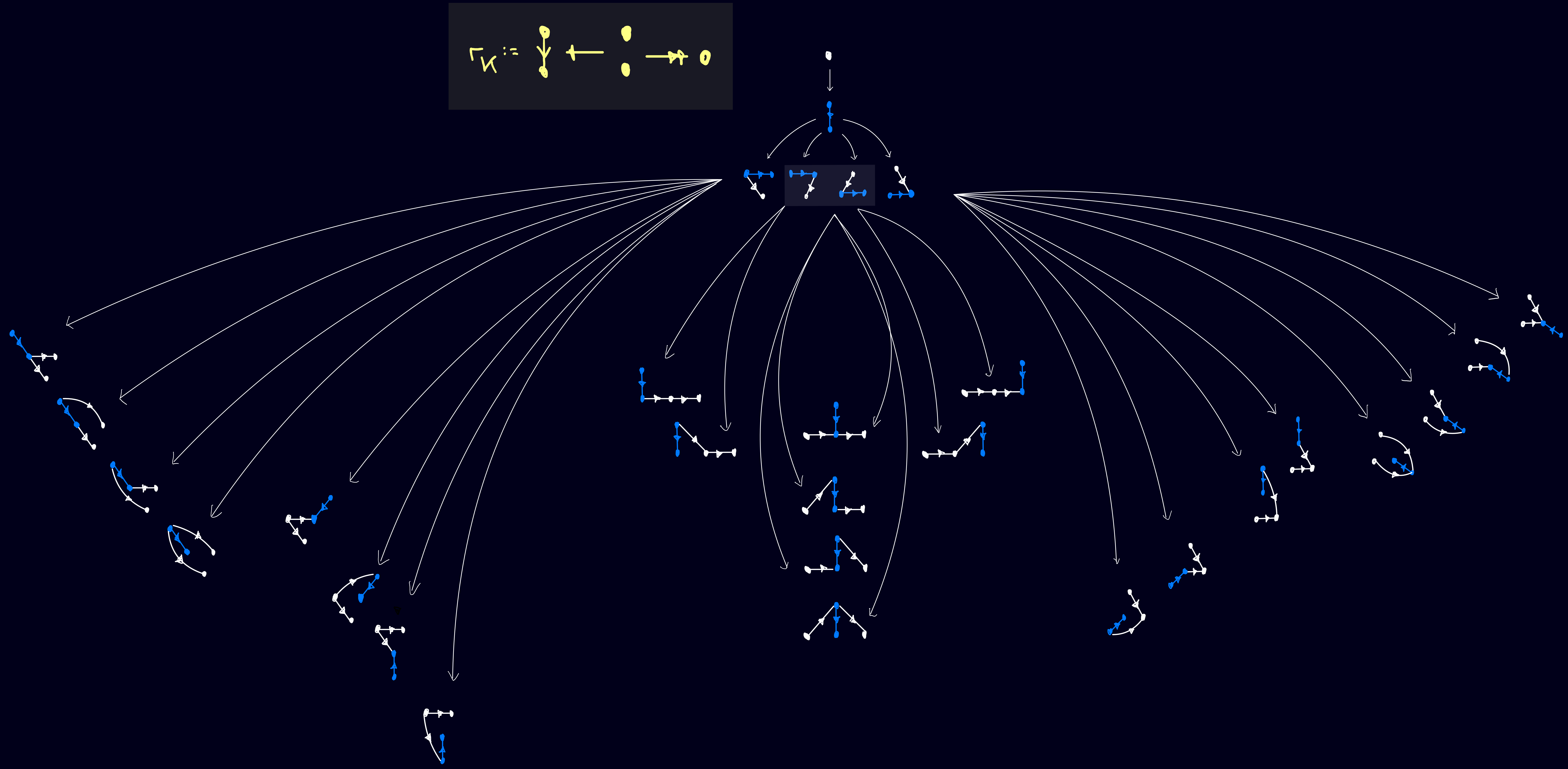
  (ii) Choice of a partition of edges incident to $m_\alpha(\bullet)$

RUNNING EXAMPLE: KONTSEVICH'S RULE

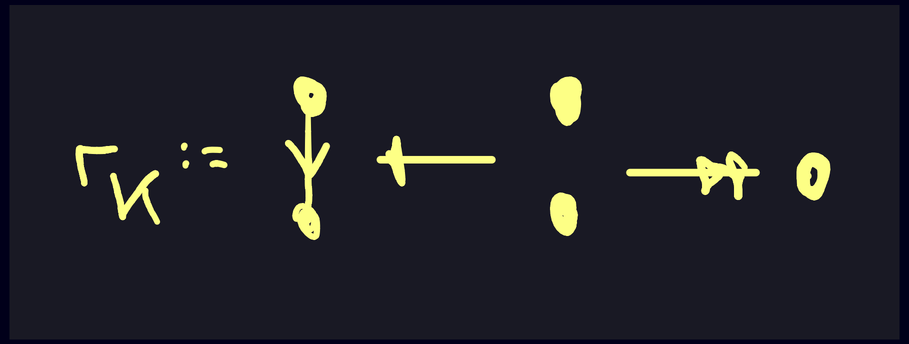




3 RUNNING EXAMPLE: KONTSEVICH'S RULE



3 Running example: Kontsevich's rule



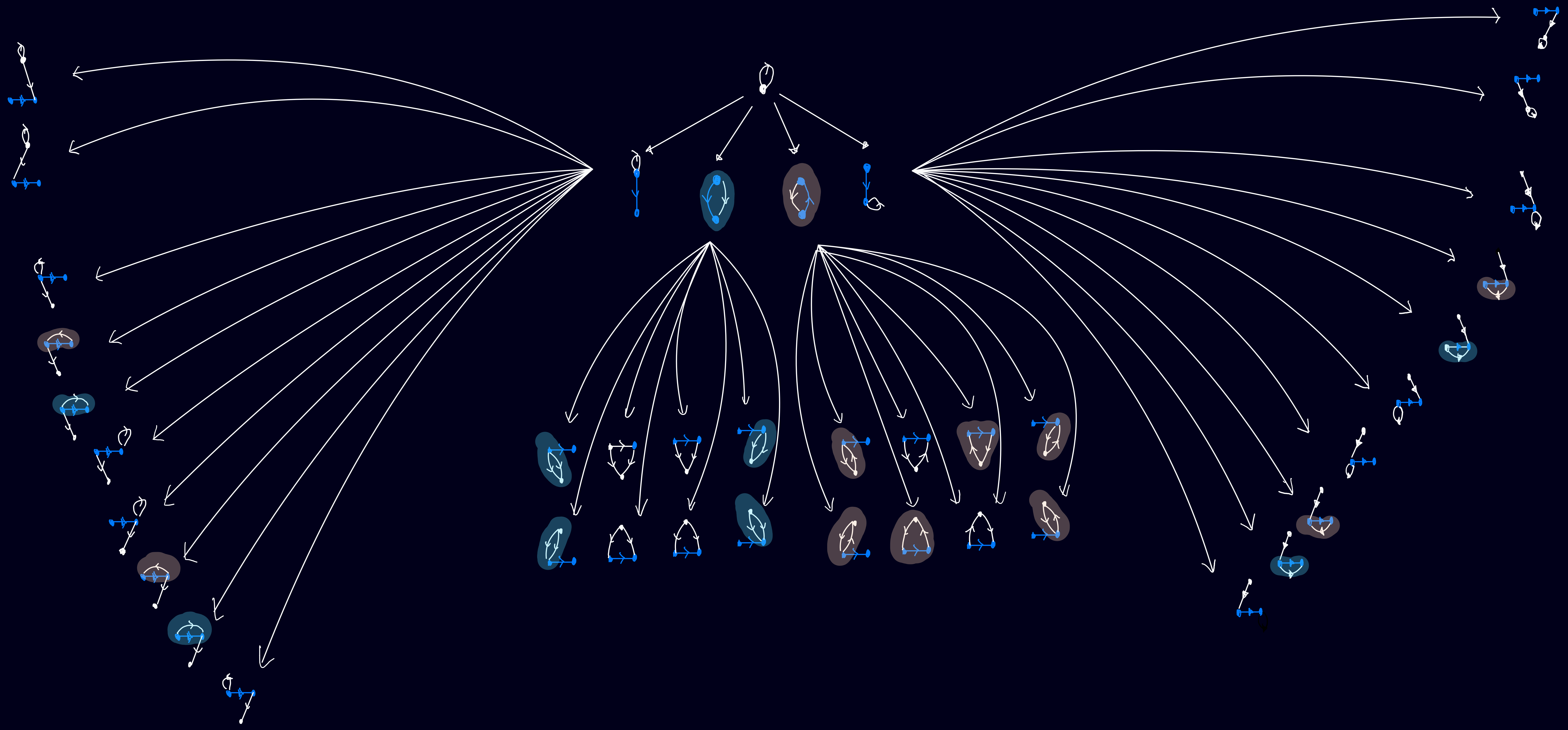
# of graphs after n applications of the rule:

1, 4, 32, 400, 6912, 153664, ...

$a_n = 2^n (n+1)^{n-2}$

OEIS A127670

3 RUNNING EXAMPLE: KONTSEVICH'S RULE



Running example: Kontsevich's rule

\# of graphs after n applications of r_u:

4, 36, 512, 10000, 248832, 7529536, …

$$a_n = 2^n (n+1)^n$$

OEIS A052746
(# of well-coloured directed trees)

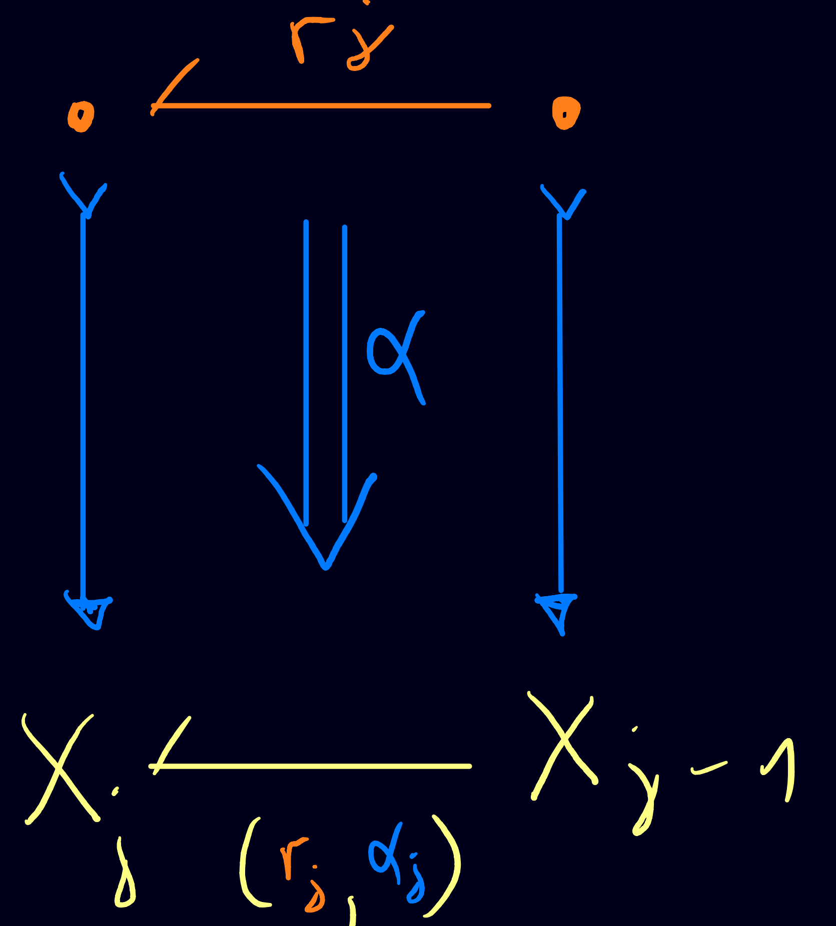
3 RUNNING EXAMPLE: KONTSEVICH'S RULE

4 COMBINATORICS & "COMPOSITIONALITY" OF REWRITING SYSTEMS

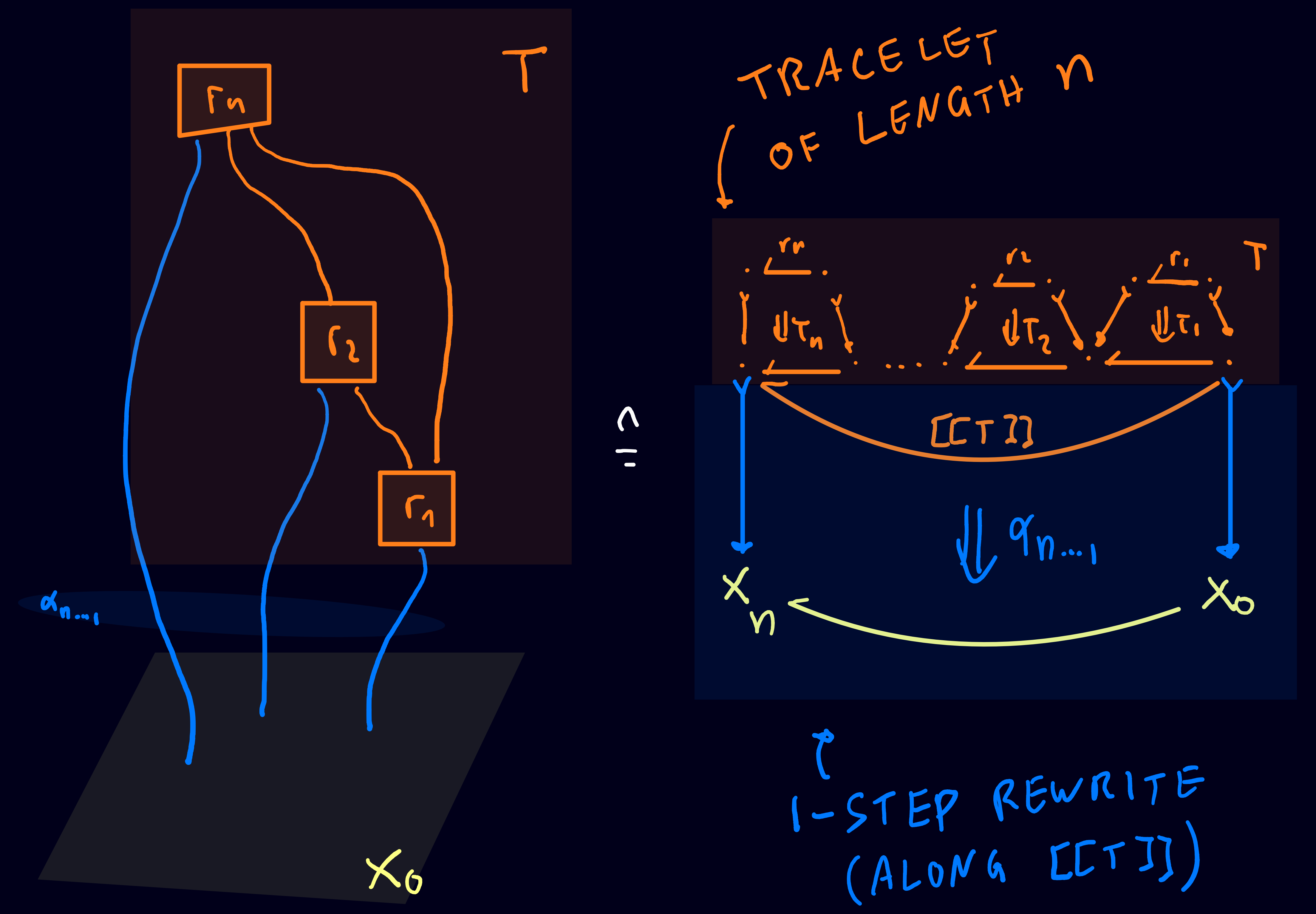
$$\begin{tikzcd}
\bullet \arrow[d] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[l, "r_j"'] \arrow[d] \\
X_j & X_{j-1} \arrow[l, "{(r_j, \alpha_j)}"]
\end{tikzcd}$$

"STATE-BASED" VIEW

4 COMBINATORICS & "COMPOSITIONALITY" OF REWRITING SYSTEMS



"STATE-BASED" VIEW



"TRANSITION-BASED" VIEW

MAIN MOTIVATION FOR "COMPOSITIONAL" REWRITING THEORY:

TRACELET HOPF ALGEBRA $\mathcal{T}$ and RULE ALGEBRA $\mathcal{R}$

capture combinatorics of INTERACTIONS between rewriting steps

(CAUSALITY!)

REPRESENTATIONS ON STATE-SPACES S

$\hat{\rho}: \mathcal{T} \to End(S)$ $\quad$ $\rho: \mathcal{R} \to End(S)$

Capture combinatorics of (distributions over) STATES of the transition system

↳ MARKOV CHAINS etc.!

ANSATZ: CATEGORIFICATION

GOAL: (i) " $\rho(s(r))|X\rangle := \sum_{\alpha} |r_{\alpha}(X)\rangle = \sum_{Y} \overbrace{\mu_{r;X}^{Y}}^{\in \mathbb{Z}_{\geq 0}} |Y\rangle$ "

ANSATZ: CATEGORIFICATION

GOAL: (i) " $\varrho(\delta(r))|X\rangle := \sum_{\alpha} |r_{\alpha}(X)\rangle = \sum_{Y} \overbrace{\mu_{r;X}{}^{Y}}^{\in \mathbb{Z}_{\geq 0}} |Y\rangle$ "

(ii) " $|X\rangle = \varrho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\varrho(\delta(r_2))\varrho(\delta(r_1)) = \sum_{r} \tau_{r_2,r_1}{}^{r}\, \varrho(\delta(r))$ "

ANSATZ: CATEGORIFICATION

GOAL:

(i) " $\varrho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \overline{\sum_{Y}} \underbrace{\mu_{r;X}{}^{Y}}_{\in \mathbb{Z}_{\geq 0}} |Y\rangle$ "

(ii) " $|X\rangle = \varrho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\varrho(\delta(r_2))\varrho(\delta(r_1)) = \overline{\sum_{r}} \tau_{r_2, r_1}{}^{r} \varrho(\delta(r))$ "

(iv) " $\varrho(\delta(r_2))\varrho(\delta(r_1)) = \varrho(\delta(r_2) \odot \delta(r_1))$ "

RULE ALGEBRA PRODUCT

ANSATZ: CATEGORIFICATION

GOAL: (i) " $\rho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \overbrace{\mu_X^Y}^{\in \mathbb{Z}_{\geq 0}} |Y\rangle$ " (ii) " $|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \sum_{r_{u}} \underbrace{T_{r_1,r_2}^{r_{u}}}_{\in \mathbb{Z}_{\geq 0}} \rho(\delta(r_{u}))$ " (iv) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$ "

$\odot$ – RULE ALGEBRA PRODUCT

I. FORMALIZE
$$\begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd}$$
AS 2-CELLS IN A DOUBLE CATEGORY

II. FORMALIZE $\mathbb{Z}_{\geq 0}$-COEFFICIENTS AS CARDINALITIES (OF SUITABLE SETS...)

METHODS: DOUBLE CATEGORIES, PRESHEAVES, FIBRATIONS, COENDS, MULTISUMS ...

Definition: A Double Category $\mathbb{D}$ is a (weakly) internal category in $\mathbf{CAT}$

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"']
\end{tikzcd}
$$

$\mathbb{D}_0$:

$$
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$\mathbb{D}_1$:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

Definition: A Double Category $\mathbb{D}$ is a (weakly) internal category in CAT

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"'] \end{tikzcd}$$

$$\mathbb{D}_0: \quad \begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

"0-cells" — objects of $\mathbb{D}_0$

"vertical morphisms" — morphisms of $\mathbb{D}_0$

$$\begin{tikzcd} O \arrow[d, "n"'] \\ Y \end{tikzcd} \quad \overset{T}{\longleftarrow\!\!\!\mid} \quad \begin{tikzcd} O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha" near end, shift right=4ex] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \quad \overset{S}{\longmapsto} \quad \begin{tikzcd} I \arrow[d, "m"] \\ X \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \quad \overset{u}{\longmapsto} \quad \begin{tikzcd} A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id" near end, shift right=4ex] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\ B & B \arrow[l, "id_B"] \end{tikzcd}$$

$$\mathbb{D}_1: \quad \begin{tikzcd} O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha" near end, shift right=4ex] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd}$$

"horizontal morphisms" — objects of $\mathbb{D}_1$

"2-cells" - morphisms of $\mathbb{D}_1$

6 DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (WEAKLY) INTERNAL CATEGORY IN $\mathbf{CAT}$

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"'] \end{tikzcd}$$

$\mathbb{D}_0$:
$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$\begin{tikzcd} O \arrow[d, "n"'] & & O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] & & I \arrow[d, "m"] \\ Y & & Y & X \arrow[l, "s"] & & X \end{tikzcd}$$
$$\xleftarrow{\ T\ } \qquad \xrightarrow{\ S\ }$$

$$\begin{tikzcd} A \arrow[d, "f"'] & \overset{U}{\longmapsto} & A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\ B & & B & B \arrow[l, "id_B"] \end{tikzcd}$$

$\mathbb{D}_1$:
$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd}$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ O' \arrow[r, phantom, ""] & I' \arrow[l, "r'"] \end{tikzcd} \quad \diamond_v \quad \begin{tikzcd} O' \arrow[d, "n'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \\ O'' & I'' \arrow[l, "r''"] \end{tikzcd} \quad := \quad \begin{tikzcd} O \arrow[d, "n' \circ n"'] \arrow[r, phantom, "\Downarrow \alpha' \circ \alpha"] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \\ O'' & I'' \arrow[l, "r''"] \end{tikzcd}$$

"vertical composition"

– composition in $\mathbb{D}_1$

$$\begin{tikzcd} C \arrow[d, "c"'] \arrow[r, phantom, "\Downarrow\alpha_2"] & B \arrow[l, "r_2"'] \arrow[d, "b"] \\ C' & B' \arrow[l, "s_2"] \end{tikzcd} \quad \diamond_h \quad \begin{tikzcd} B \arrow[d, "b"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & A \arrow[l, "r_1"'] \arrow[d, "a"] \\ B' & A' \arrow[l, "s_1"] \end{tikzcd} \quad := \quad \begin{tikzcd} C \arrow[d, "c"'] \arrow[r, phantom, "\Downarrow \alpha_2 \diamond_h \alpha_1"] & A \arrow[l, "r_2 \diamond_h r_1"'] \arrow[d, "a"] \\ C' & A' \arrow[l, "s_2 \diamond_h s_1"] \end{tikzcd}$$

"horizontal composition" ← IN GENERAL ONLY WEAKLY ASSOCIATIVE!

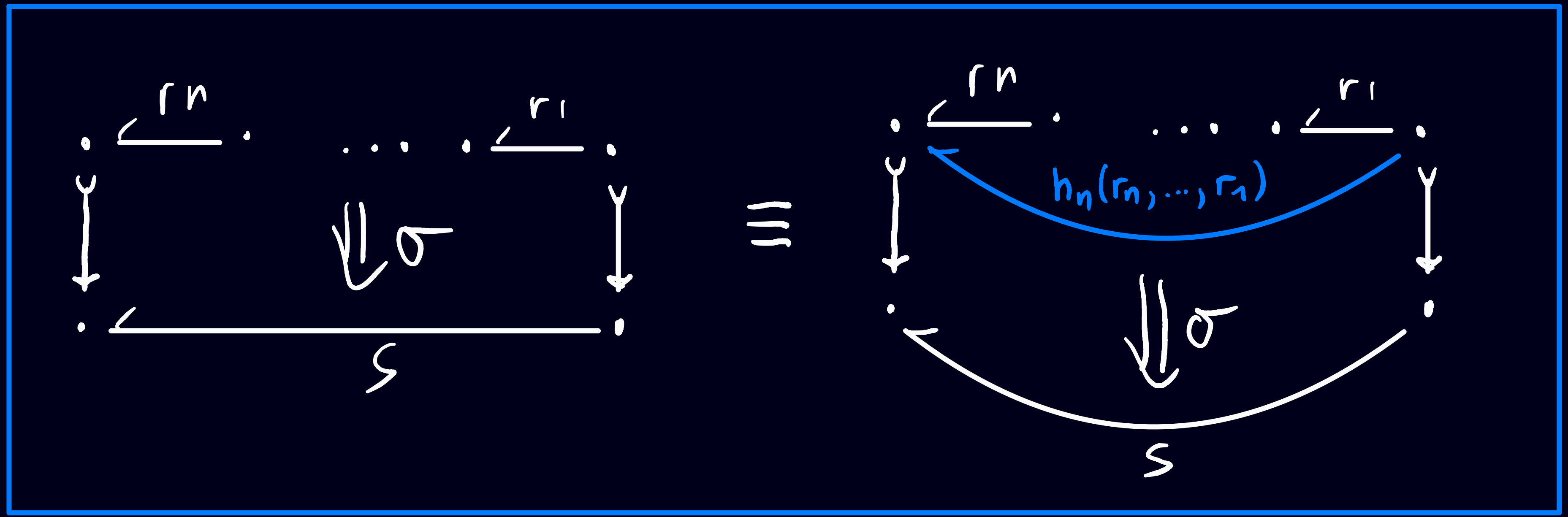
7 DEFINITION: A PRESENTATION OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$

OF FUNCTORS $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2,-_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2: \quad h_{n+1}(-_{n+1},\dots,-_1) \cong h_2(-_{n+1}, h_n(-_n,\dots,-_1))$$

▸ NOTATIONAL CONVENTION:



7 DEFINITION: A **PRESENTATION** OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$

OF FUNCTORS $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2,-_1) := -_2 \diamond_h -_1,$$

$$\forall n\geq 2: \; h_{n+1}(-_{n+1},\dots,-_1) \cong h_2(-_{n+1}, h_n(-_n,\dots,-_1))$$

- NOTATIONAL CONVENTION:

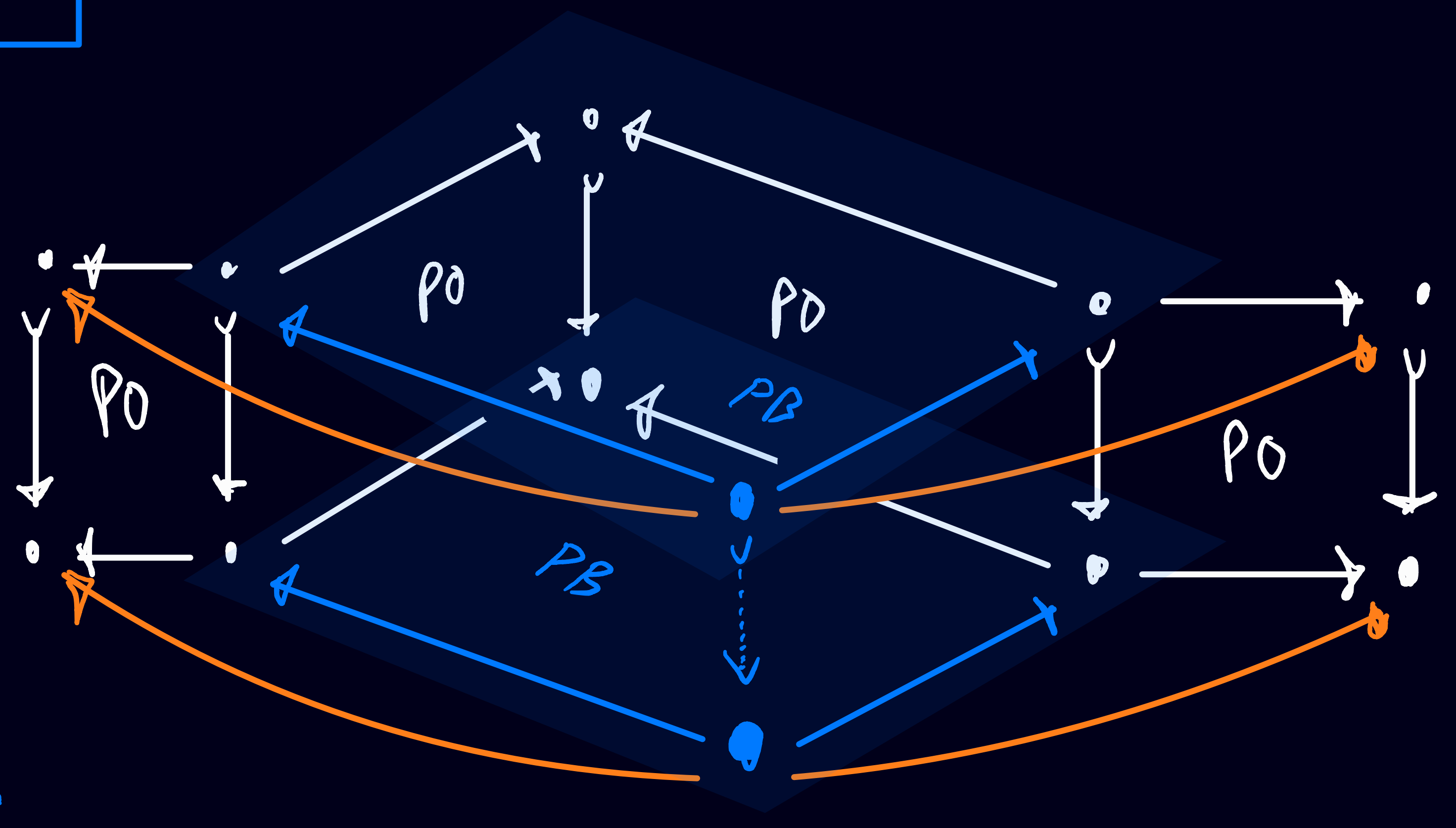
$$\begin{tikzcd} \bullet & \bullet \arrow[l, "r_n"'] & \cdots & \bullet & \bullet \arrow[l, "r_1"'] \\ \bullet \arrow[from=u, rightarrowtail] & & \arrow[u, phantom, "\Downarrow\sigma"] & & \bullet \arrow[from=u, rightarrowtail] \arrow[llll, "s"] \end{tikzcd} \equiv \begin{tikzcd} \bullet & \bullet \arrow[l, "r_n"'] & \cdots & \bullet & \bullet \arrow[l, "r_1"'] \arrow[llll, bend left, "h_n(r_n,\dots,r_1)"] \\ \bullet \arrow[from=u, rightarrowtail] & & \arrow[u, phantom, "\Downarrow\sigma"] & & \bullet \arrow[from=u, rightarrowtail] \arrow[llll, bend left, "s"] \end{tikzcd}$$

- EXAMPLE:

$$\begin{tikzcd} O \arrow[d, rightarrowtail, "n"'] \arrow[r, phantom, ""] & I \arrow[l, "r"'] \arrow[d, rightarrowtail, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} := \begin{tikzcd} O \arrow[d, rightarrowtail, "n"'] & K_r \arrow[l, rightarrowtail, "o_r"'] \arrow[r, "i_r"] \arrow[d, rightarrowtail, "k_\alpha"] \arrow[dl, phantom, "PO"] & I \arrow[d, rightarrowtail, "m"] \arrow[dl, phantom, "PO"] \\ Y & K_s \arrow[l, rightarrowtail, "o_s"] \arrow[r, "i_s"'] & X \end{tikzcd}$$

$\rightarrow$ HORIZONTAL COMPOSITION:

$\hat{=}$ CHOICE OF PULLBACKS (PBs)!



8 KEY CONCEPT: COVARIANT PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ <u>IDEA:</u> $\forall r \in \mathbb{D}_1 : \quad \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r(Y \xleftarrow{s} X)\right| = \left| \left\{ \begin{array}{ccc} O & \xleftarrow{r} & I \\ n \downarrow & \Downarrow\alpha & \downarrow m \\ Y & \xleftarrow{s} & X \end{array} \in \mathbb{D}_1 \right\} \right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

▸ BUT: we want "$\varrho(\delta(r))|x\rangle = \underbrace{\varrho(\delta(r))\,\varrho(\delta(\overbrace{x \leftarrow \emptyset}^{?}))|\overbrace{\emptyset}^{?}\rangle}_{?} = \sum_{\alpha} |r_\alpha(x)\rangle = \sum_{y} \underbrace{\overbrace{M_{r,x}^{\;y}}^{\in \mathbb{Z}_{\geq 0}}}_{?} |y\rangle$"

8 KEY CONCEPT: COVARIANT PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

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▸ BUT: we want "$\varrho(\delta(r))|X\rangle = \underbrace{\varrho(\delta(r))\,\varrho(\delta(X \overbrace{\leftarrow}^{?} \emptyset))}_{?}|\overset{?}{\emptyset}\rangle = \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\overbrace{M_{r,X}^{Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?} |Y\rangle$"

▸ ASSUMPTION: $\mathbb{D}_0$ HAS A STRICT INITIAL OBJECT $\emptyset$ (i.e., $\forall\, X \in \mathbb{D}_0 : \exists!\, \emptyset \rightarrowtail X \wedge \forall\, X \xrightarrow{f} \emptyset : X = \emptyset$),

AND SUCH THAT (i) $\forall\, X \in \mathbb{D}_0 : \exists!\, (X \leftarrow \emptyset) \in ob(\mathbb{D}_1) \wedge \exists!\, (\emptyset \leftarrow X) \in ob(\mathbb{D}_1)$

(ii) $\forall\, {\scriptstyle f}\!\downarrow^{X}_{Y} \in \mathbb{D}_0 : \left| \left\{ \begin{array}{ccc} X & \leftarrow & \emptyset \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \| \\ Y & \leftarrow & \emptyset \end{array} \in \mathbb{D}_1 \right\} \right| \leq 1 \wedge \left| \left\{ \begin{array}{ccc} \emptyset & \leftarrow & X \\ \| & \Downarrow\beta & \downarrow{\scriptstyle f} \\ \emptyset & \leftarrow & Y \end{array} \in \mathbb{D}_1 \right\} \right| \leq 1$

9 DEFINITION: A COEND FOR A FUNCTOR $F : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Set}}$ IS DEFINED AS

$$\int^{C \in \mathcal{C}} F(C,C) = \left(\coprod_{C \in \mathcal{C}} F(C,C) \right) \Big/ \sim$$

with: $\underset{\in F(C,C)}{(C,x)} \sim \underset{\in F(C',C')}{(C',x')} :\Leftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C',C) :\ x = F(\gamma, id)y \ \wedge\ x' = F(id,\gamma)y$

9 DEFINITION: A COEND FOR A FUNCTOR $F : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathbf{Set}}$ IS DEFINED AS

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with: $(\underset{\overset{\cup}{F(C,C)}}{(C}, x) \sim (\underset{\overset{\cup}{F(C',C')}}{C'}, x') :\Leftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C',C) : x = F(\gamma, id)y \ \wedge\ x' = F(id, \gamma)y$

KEY CONCEPT: CONVOLUTION PRODUCTS OF PRESHEAVES $F_n, \ldots, F_1 : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

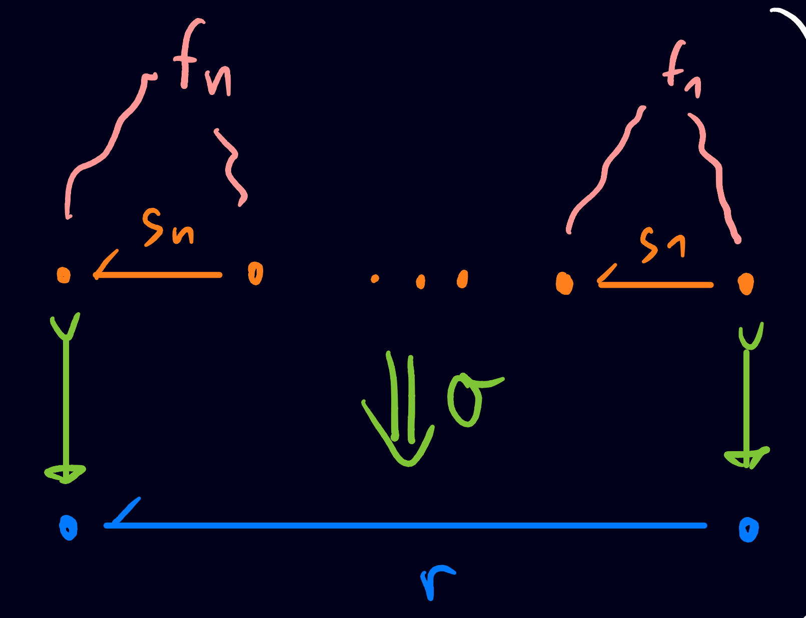
$$(F_n * \cdots * F_1) := r \mapsto \int^{S = (S_n, \ldots, S_1) \in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S)$$

- $\mathbb{D}_1(h_n(-), r) : \mathbb{D}_n^{op} \to \underline{\mathbf{Set}}$
- $\mathbb{F}_n := F_n \times \ldots \times F_1 : \mathbb{D}_n \to \underline{\mathbf{Set}}$

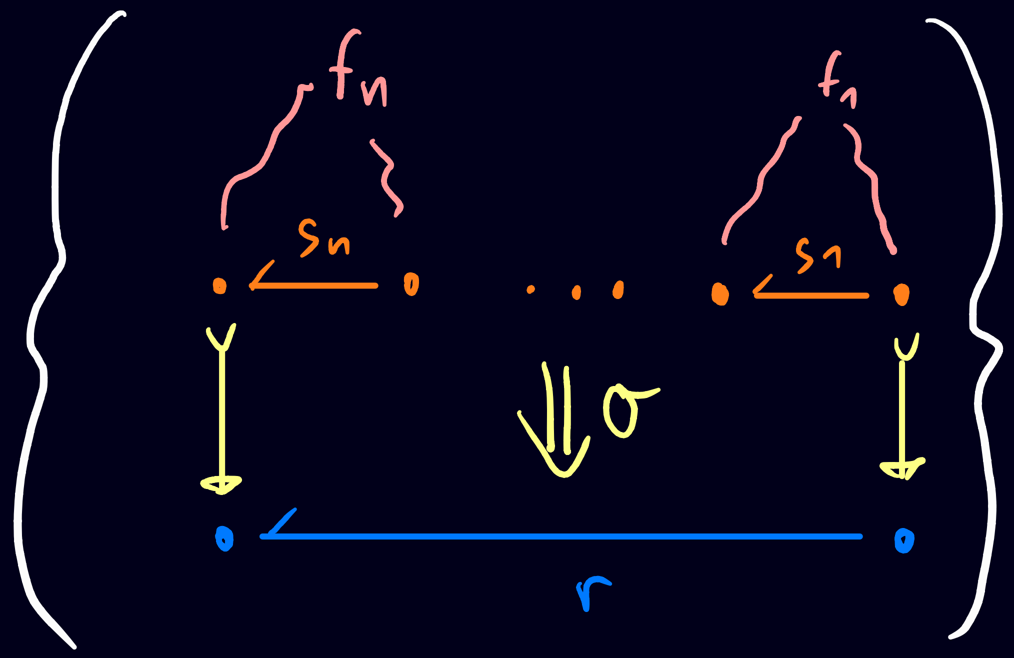
$\cong$

$Lan_{h_n}(\mathbb{F}_n)$

$$= \left\{ (S, (\sigma, f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim$$

$\equiv \Big\{$  $\Big\} \Big/ \sim$

$$(F_n * \cdots * F_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\equiv$$

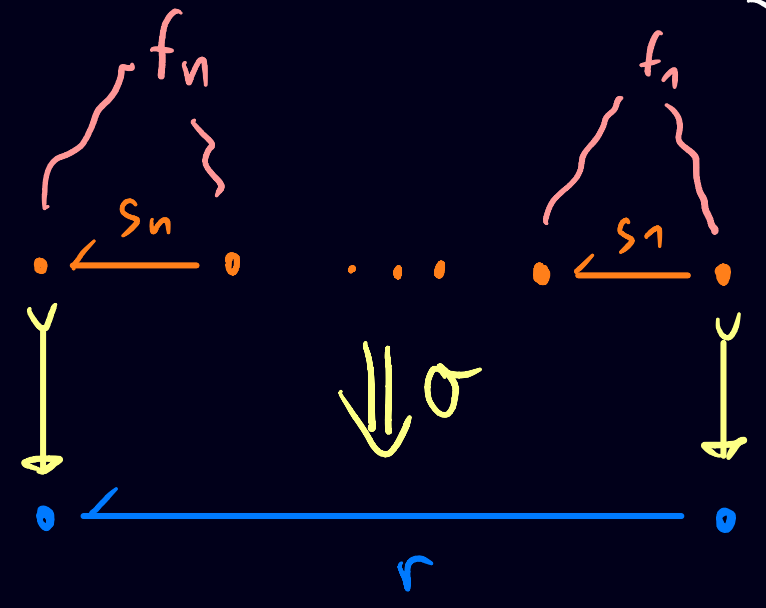


$/\sim$

- $(S,(\sigma,f)) \sim (S',(\sigma',f')) \;:\Leftrightarrow\; \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'),r) \times \mathbb{F}_n(S):$

$$(\sigma,f) = (\mathbb{D}_1(h_n(A),r)\tau,\ g) \;\wedge\; (\sigma',f') = (\tau,\ \mathbb{F}_n(A)g)$$

$$(\mathbb{F}_n * \cdots * \mathbb{F}_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\equiv\; \Bigg\{$$

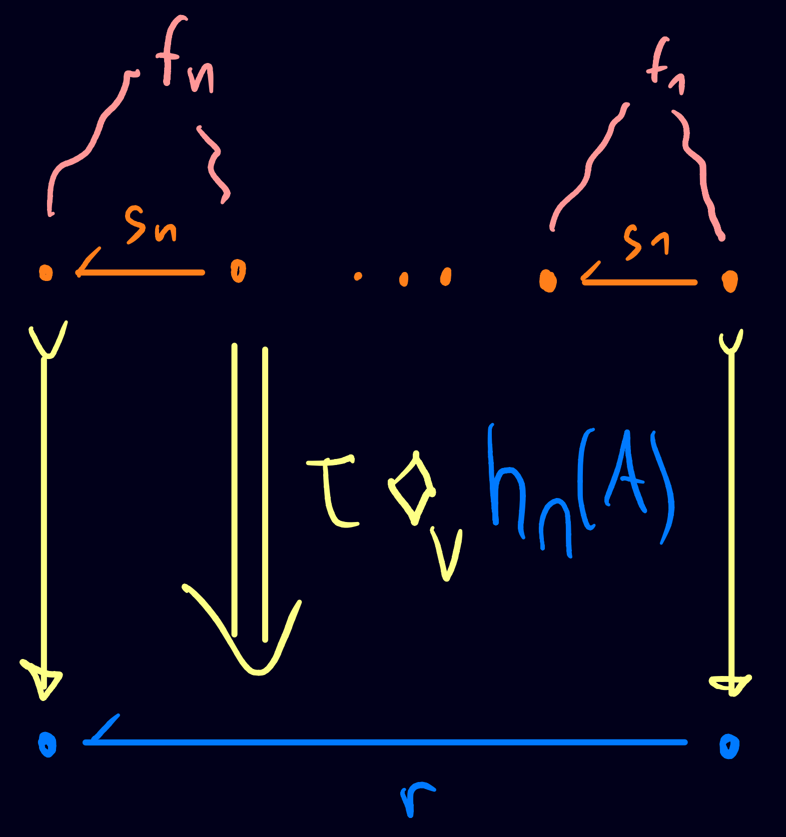


$$\Bigg\} \Big/ \sim$$

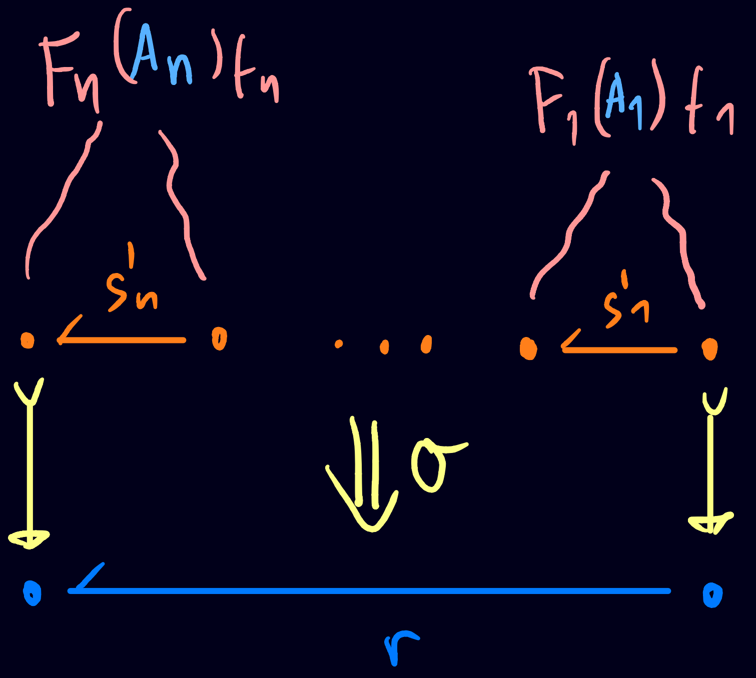
$$\cdot\ (S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$$

$$(\sigma,f) = \big(\mathbb{D}_1(h_n(A), r)\tau,\ g\big) \ \wedge\ (\sigma',f') = \big(\tau,\ \mathbb{F}_n(A)g\big)$$

$\hookrightarrow\ \forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:

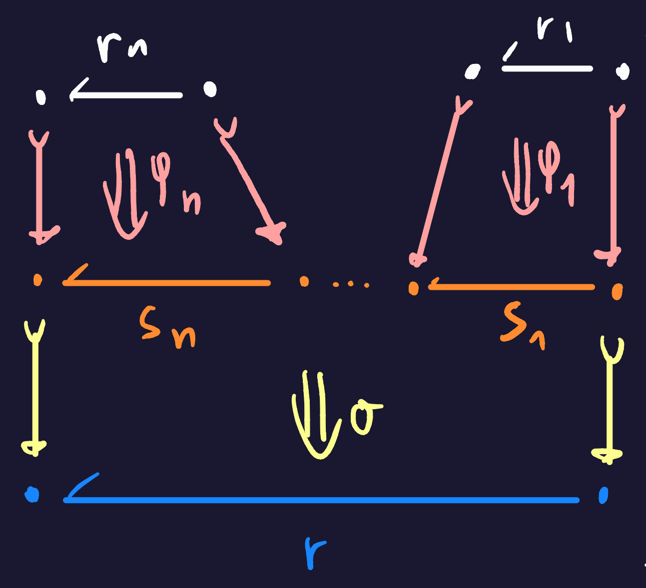


$\sim$



EXAMPLE: $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -) : \mathbb{D}_1 \to \underline{\mathbf{Set}}$ $(j = 1, \ldots, n)$

$\hookrightarrow\ (\hat{\Delta}_{r_n} * \cdots * \hat{\Delta}_{r_1})(r) = \Bigg\{$



$\Bigg\} \Big/ \sim$

Key Concept: Fibrational Structures

- Definition: A functor $G : \mathcal{E} \to \mathcal{B}$ is a **Grothendieck opfibration** iff

$$
\forall\;
\begin{tikzcd}
e \arrow[d, mapsto, "G"'] & \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\; \exists\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] & e' \arrow[d, mapsto, "G"] \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\; \forall\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "G"'] & e' \arrow[d, mapsto, "G"] & e'' \arrow[d, mapsto, "G"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon) = g \circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "G"'] & e' \arrow[r, dashed, "\beta", "\exists!"'] \arrow[d, mapsto, "G"] & e'' \arrow[d, mapsto, "G"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon) = g \circ f"'] & b' \arrow[r, "g = G(\beta)"'] & b''
\end{tikzcd}
$$

KEY CONCEPT: FIBRATIONAL STRUCTURES

- DEFINITION: A FUNCTOR $G : \mathcal{E} \to \mathcal{B}$ IS A GROTHENDIECK OPFIBRATION IFF

$$
\forall\ \begin{tikzcd} e \arrow[d, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}
\ :\ \exists\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] & e' \arrow[d, "G"] \\ b \arrow[r, "f"'] & b' \end{tikzcd}
\ :\ \forall\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, "G"] & e'' \arrow[d, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd}
\ :\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, "G"] \arrow[r, dashed, "\beta"', "\exists!"] & e'' \arrow[d, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b'' \end{tikzcd}
$$

- DEFINITION: A FUNCTOR $S : \mathcal{E} \to \mathcal{B}$ IS A STREET OPFIBRATION IFF

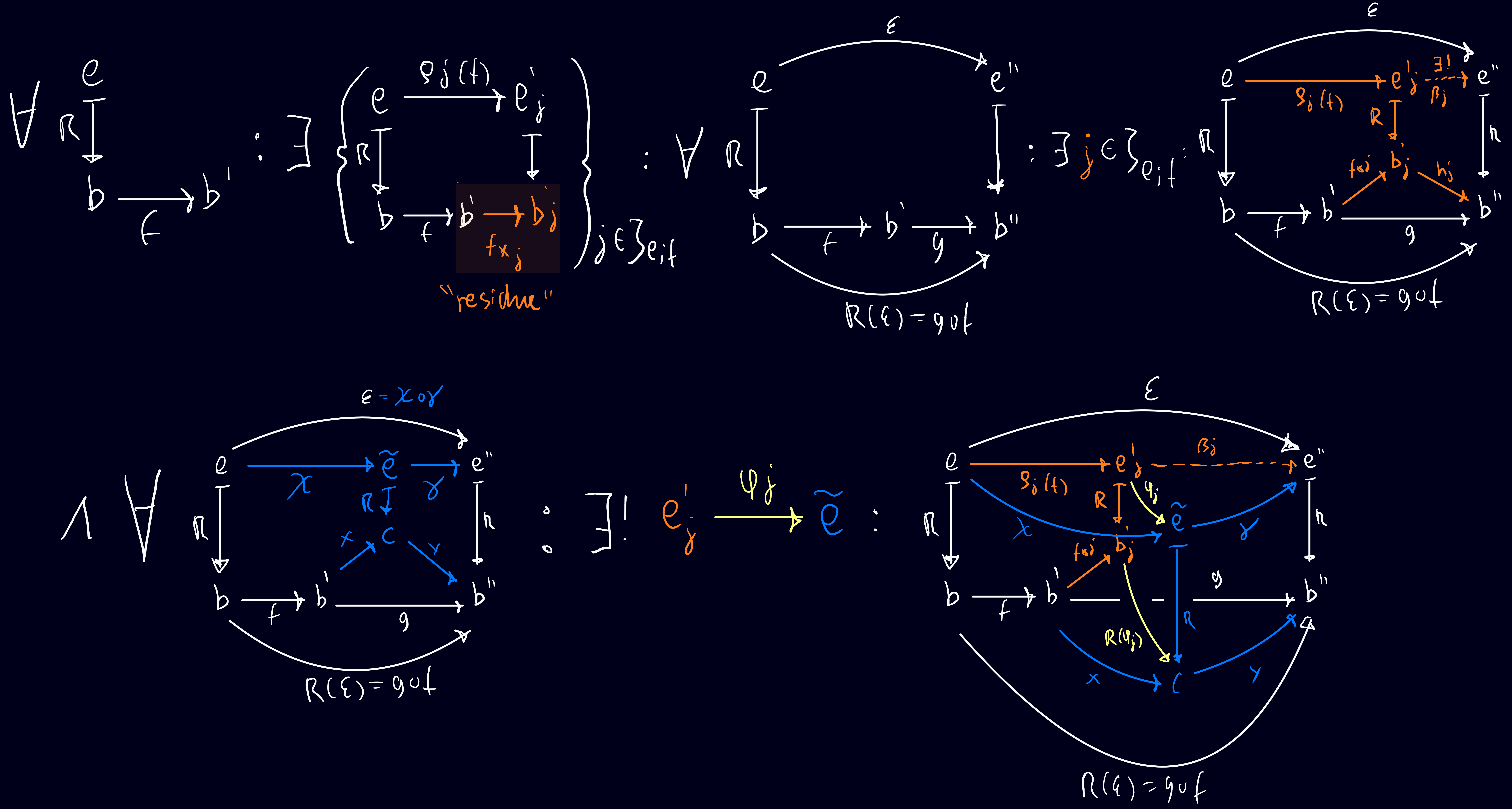
$$
\forall\ \begin{tikzcd} e \arrow[d, "S"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}
\ :\ \exists\ \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] & & \tilde{e} \arrow[d, "S"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd}
\ :\ \forall\ \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] \arrow[rrr, bend left, "\varepsilon"] & & \tilde{e} \arrow[d, "S"] & e'' \arrow[d, "S"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g"'] & b'' \end{tikzcd}
\ :\ \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] \arrow[rrr, bend left, "\varepsilon"] & & \tilde{e} \arrow[d, "S"] \arrow[r, dashed, "\beta"] & e'' \arrow[d, "S"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g=S(\beta)"'] & b'' \end{tikzcd}
$$

11 DEFINITION: A FUNCTOR $M:\mathcal{E}\to\mathcal{B}$ IS A MULTI-OPFIBRATION IFF

$$
\forall\ \begin{tikzcd} e \arrow[d, mapsto, "M"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}
\ :\ \exists \left\{ \begin{tikzcd} e \arrow[r, "\mu_j(f)"] \arrow[d, mapsto, "M"'] & e_j \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \right\}_{j\in J_{e,f}}
\ :\ \forall\ \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & & e'' \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd}
\ :\ \exists\, j\in J_{e,f} :\ \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[r, "\mu_j(f)"'] \arrow[d, mapsto, "M"'] & e'_j \arrow[r, dashed, "\beta_j"] \arrow[r, phantom, "\exists!"', shift right=2] \arrow[d, mapsto, "M"] & e'' \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=M(\beta_j)"'] & b'' \end{tikzcd}
$$

$$
\wedge\ \forall\ \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[r, "x"] \arrow[d, mapsto, "M"'] & X \arrow[r, "\gamma"] \arrow[d, mapsto, "M"] & e'' \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd}
\ :\ \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[r, "\mu_j(f)"'] \arrow[dr, bend right, "x"'] \arrow[dd, mapsto, "M"'] & e'_j \arrow[r, dashed, "\beta_j"] \arrow[d, dashed, "\exists!"] \arrow[dd, mapsto, "M", shift left=4] & e'' \arrow[dd, mapsto, "M"] \\ & X \arrow[ur, "\gamma"'] \arrow[d, mapsto, "M"] & \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd}
$$

DEFINITION: A FUNCTOR $R : \mathcal{E} \to \mathcal{B}$ IS A RESIDUAL MULTI-OPFIBRATION IFF



DEFINITION: LET $X : \mathcal{E} \to \mathcal{B}$ BE AN $\mathbb{X}$-OPFIBRATION $(\mathbb{X} \in \{G, S, M, R\})$.

THEN A CLEAVAGE FOR X IS DEFINED AS A CHOICE OF REPRESENTATIVE FOR EACH $\mathbb{X}$-OPCARTESIAN LIFTING:

$$G^{*}\left(\begin{tikzcd} e \arrow[d, mapsto, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[r, "G^{*}(f)"] \arrow[d, mapsto, "G"'] & b \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] & b \end{tikzcd}$$

$$S^{*}\left(\begin{tikzcd} e \arrow[d, mapsto, "s"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[rr, "\sigma^{*}(f)"] \arrow[d, mapsto, "s"'] & & \tilde{e} \arrow[d, mapsto, "s"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd}$$

$$M^{*}\left(\begin{tikzcd} e \arrow[d, mapsto, "m"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[r, "\mu_j^{*}(f)"] \arrow[d, mapsto, "m"'] & e'_j \arrow[d, mapsto, "m"] \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right\}_{j \in \mathfrak{Z}^{*}_{e;f}}$$

$$R^{*}\left(\begin{tikzcd} e \arrow[d, mapsto, "R"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[rr, "\rho_j^{*}(f)"] \arrow[d, mapsto, "R"'] & & e'_j \arrow[d, mapsto, "R"] \\ b \arrow[r, "f"'] & b' \arrow[r, "fx_j"'] & b'_j \end{tikzcd}\right\}_{j \in \mathfrak{Z}^{*}_{e;f}}$$

ONE REPRESENTATIVE PER EQUIVALENCE CLASS IN $\mathfrak{Z}_{e;f}$!

13 EMPIRICAL RESULT: $\mathbb{D}$ FOR COMPOSITIONAL* REWRITING SEMANTICS

*2204.07175

- $h_2 = \diamond_n : \mathbb{D}_2 \longrightarrow \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION, i.e.,

$$\forall \quad \begin{tikzcd} R=(r_2,r_1) \arrow[d, "h_2"'] & \\ r \arrow[r, "\alpha"'] & s \end{tikzcd} \quad : \exists \quad \begin{tikzcd} R \arrow[rr, "A"] \arrow[d, "h_2"'] & & T \arrow[d, "h_2"] \\ r \arrow[r, "\alpha"'] & s \arrow[r, "\cong", "\varphi_\alpha"'] & t \end{tikzcd} \quad : S(\varphi_\alpha) = id_{S(s)} \wedge T(\varphi_\alpha) = id_{T(s)} \wedge$$

(STREET OPFIBRATION CONDITIONS)

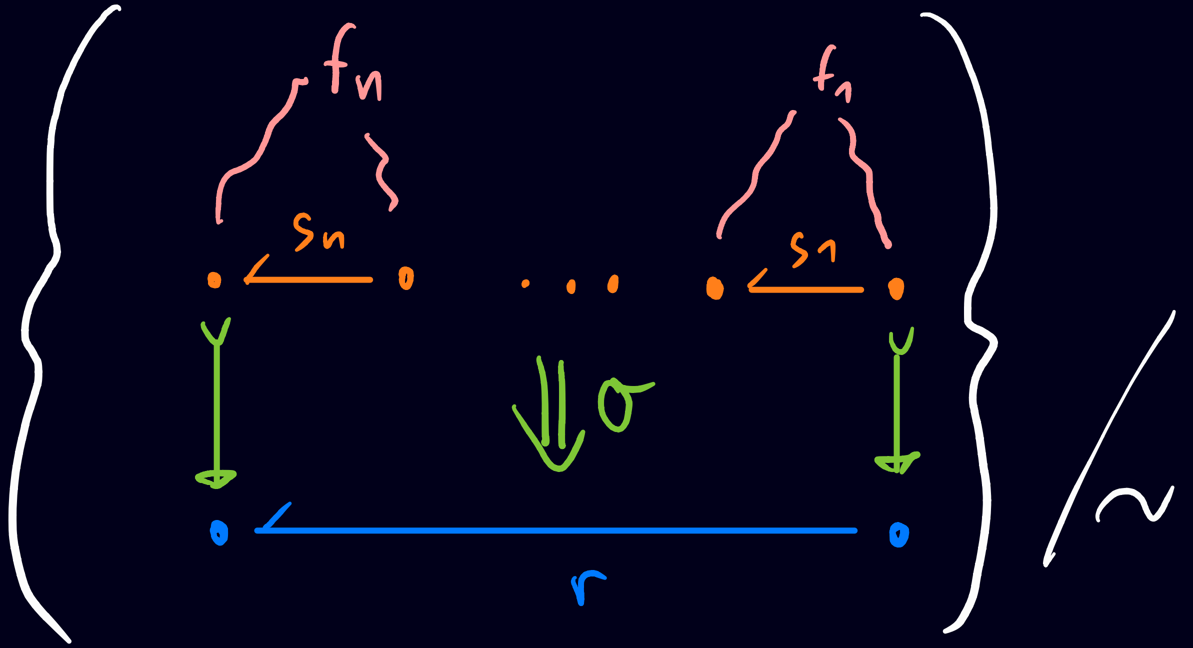
$$\forall \quad \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_2"'] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet \arrow[rr, phantom, "\Downarrow\alpha"] & & \bullet \arrow[ll, "s"] \end{tikzcd} \quad : \exists \quad \begin{tikzcd} \bullet \arrow[d] \arrow[r, phantom, "\Downarrow A_2"] & \bullet \arrow[l, "r_2"'] \arrow[d] \arrow[r, phantom, "\Downarrow A_1"] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet & \bullet \arrow[l, "t_2"'] \arrow[phantom, d, "\cong\ \Downarrow \varphi_\alpha^{-1}"] & \bullet \arrow[l, "t_1"'] \arrow[ll, bend left, "s"] \\ & & \end{tikzcd} \quad : \varphi_\alpha^{-1} \diamond_v (A_2 \diamond A_1) = \alpha$$

"GLOBULAR" ISOMORPHISM

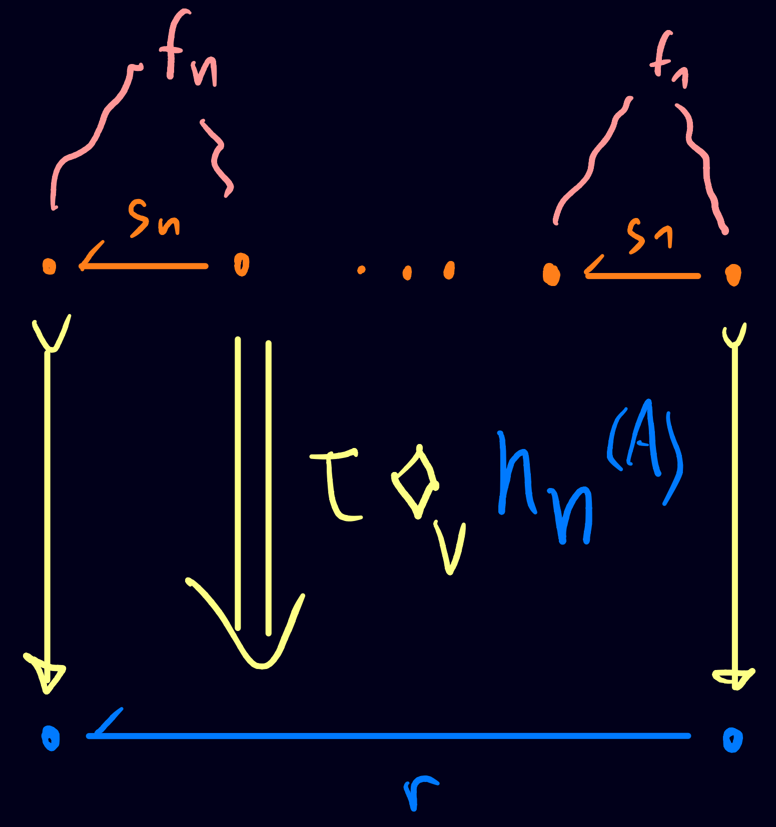
BY INDUCTION ON n, ONE FINDS THAT

$\forall n \geq 2 : h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$ ARE "GLOBULAR" STREET OPFIBRATIONS

CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) =$ 

$\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



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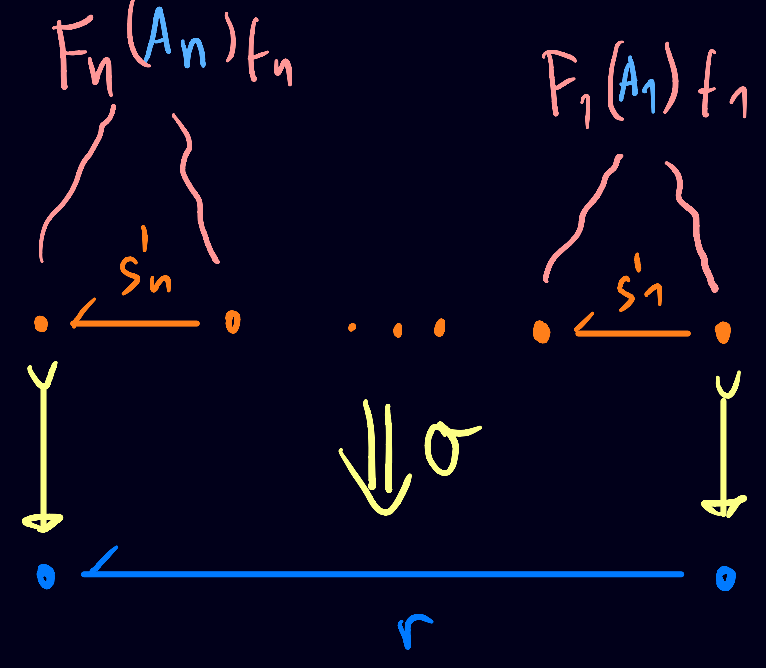
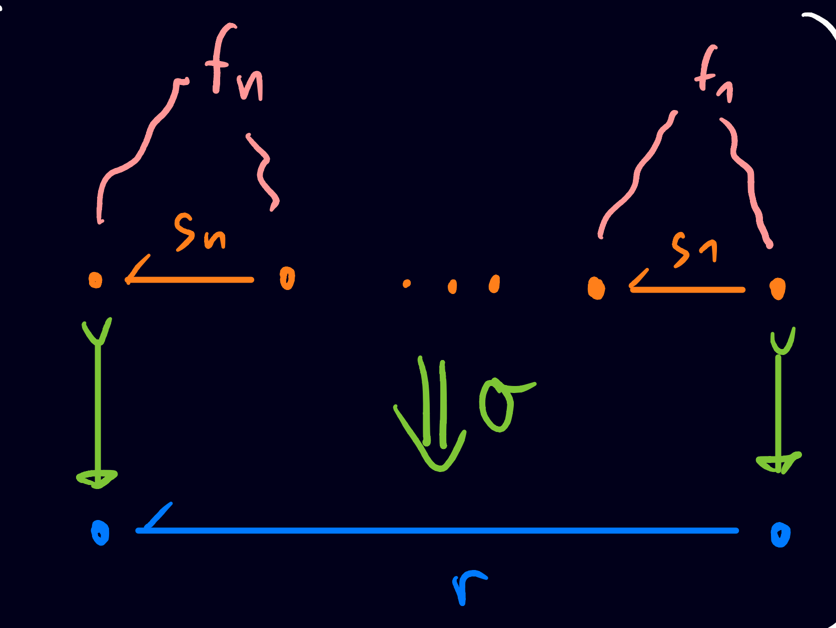
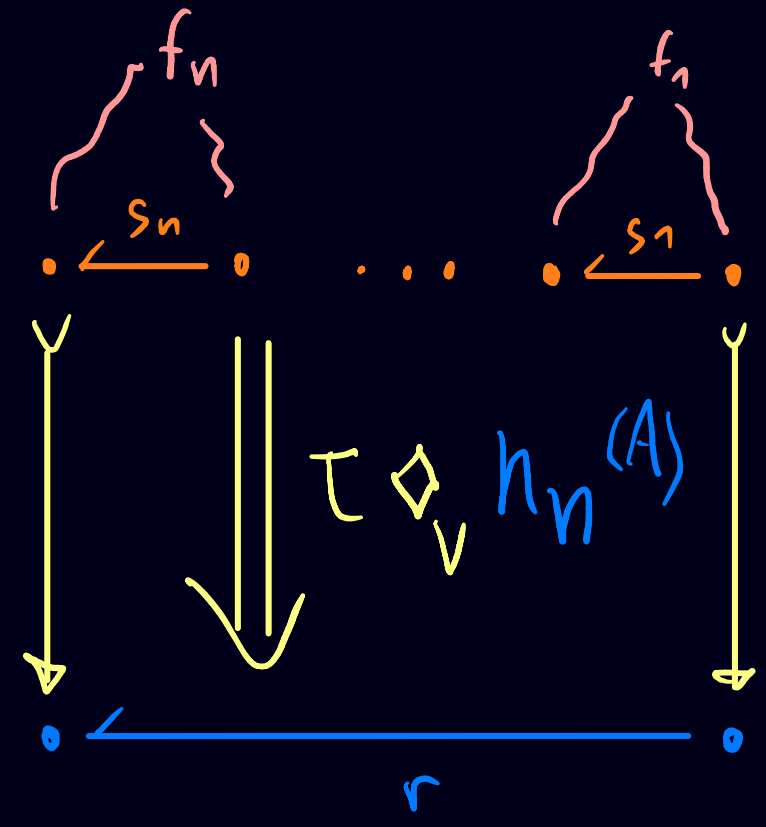


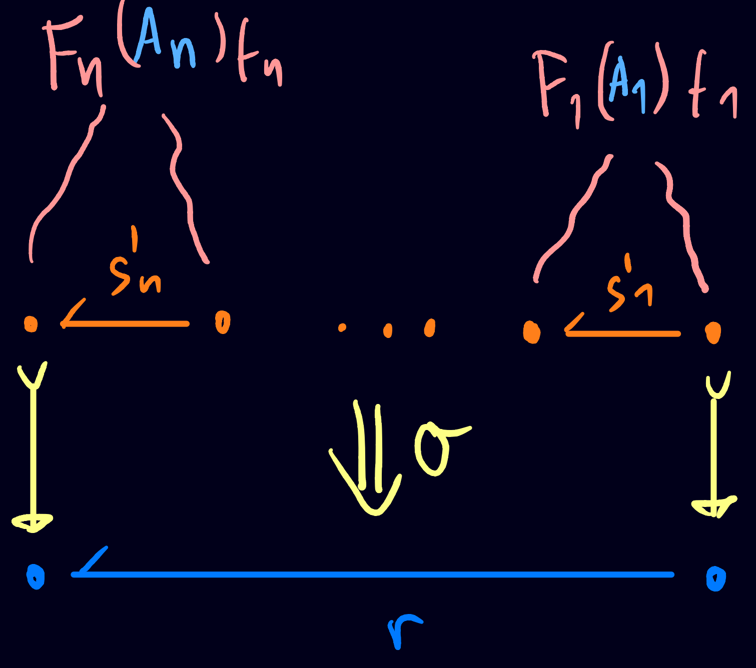
CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) = \left\{ \right.$  $\left. \right\} / \sim$

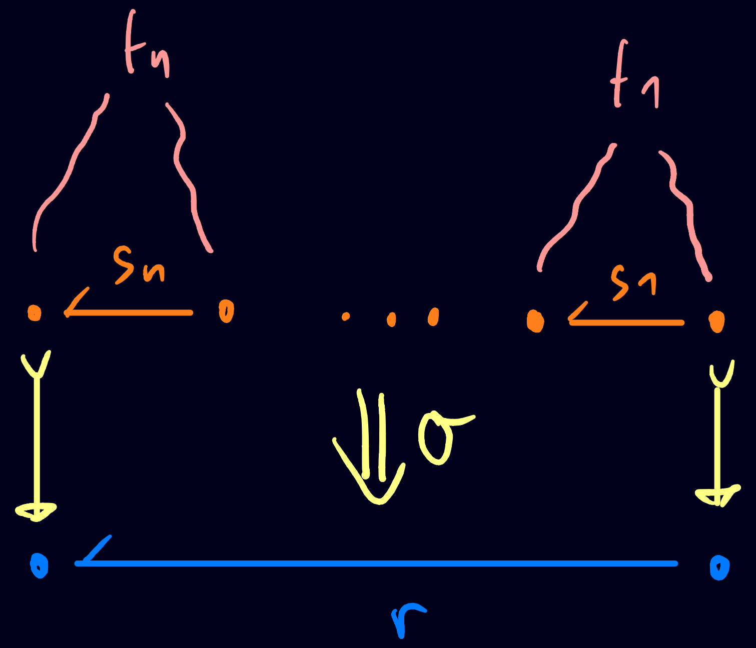
$\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



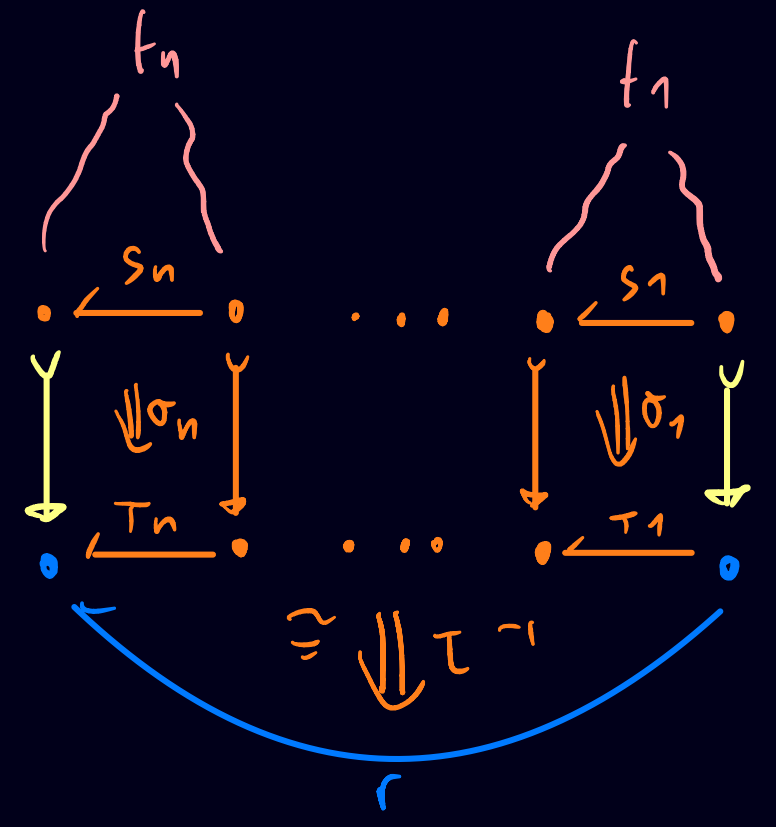
$\sim$



<u>NOW</u>: $h_n : \mathbb{D}_n \to \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION $\Rightarrow \forall$



: $\exists$



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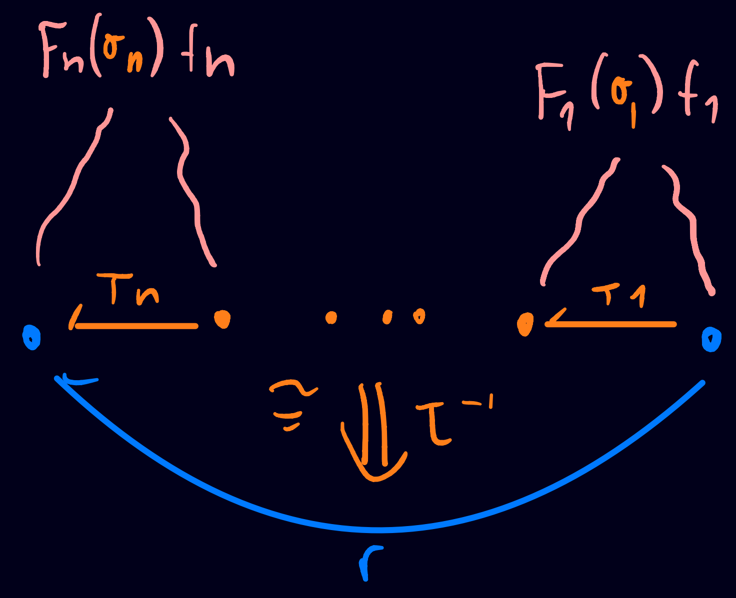
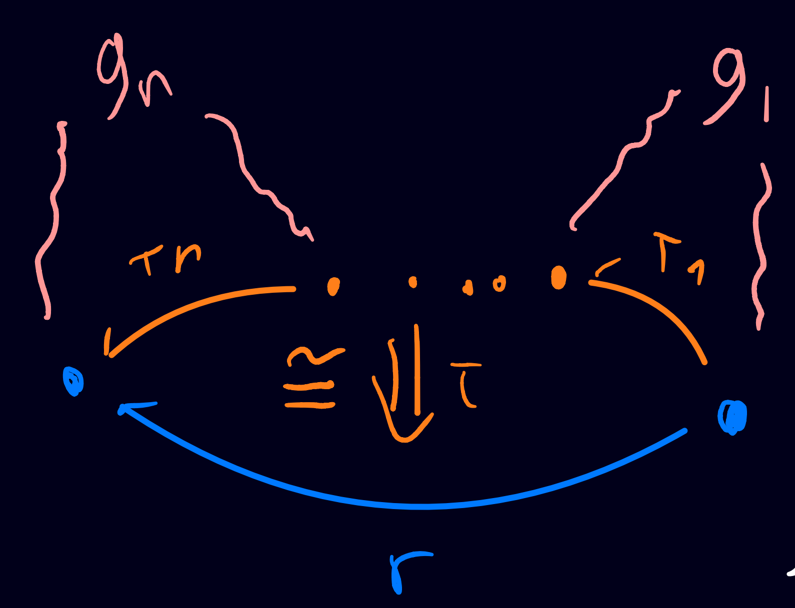
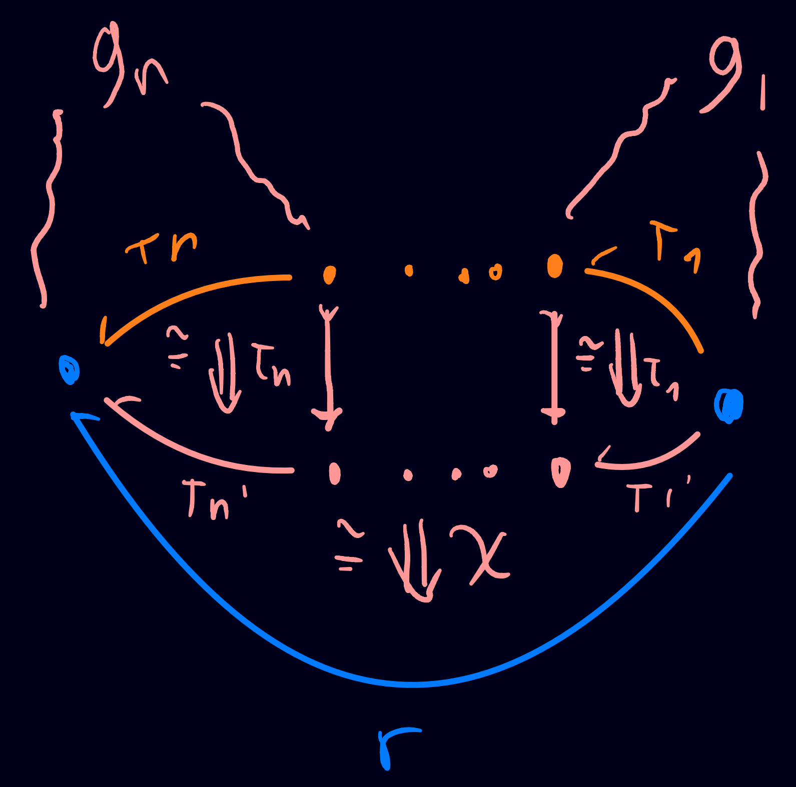
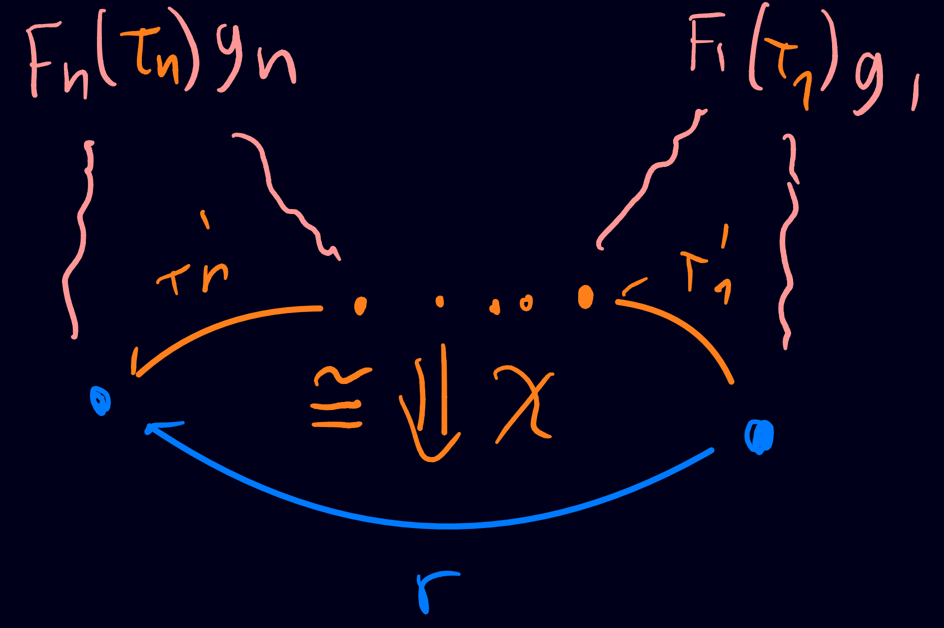
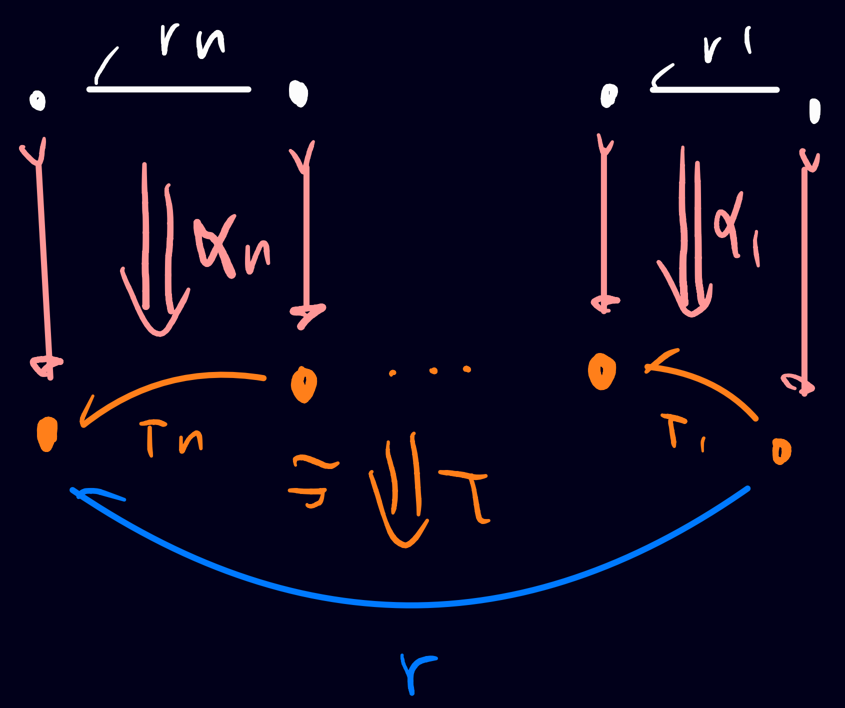


↳ $(F_n * \dots * F_1)(r) \cong \Big\{$  $\Big\} / \cong_g$

WHERE  $\cong_g$ 

EXAMPLE: FOR $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -)$ $(j = 1, \dots, n)$

$(\hat{\Delta}_{r_n} * \dots * \hat{\Delta}_{r_1})(r) \cong \Big\{$  $\Big\} / \cong_g$

KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}}\,,\quad r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

PROOF (SKETCH):

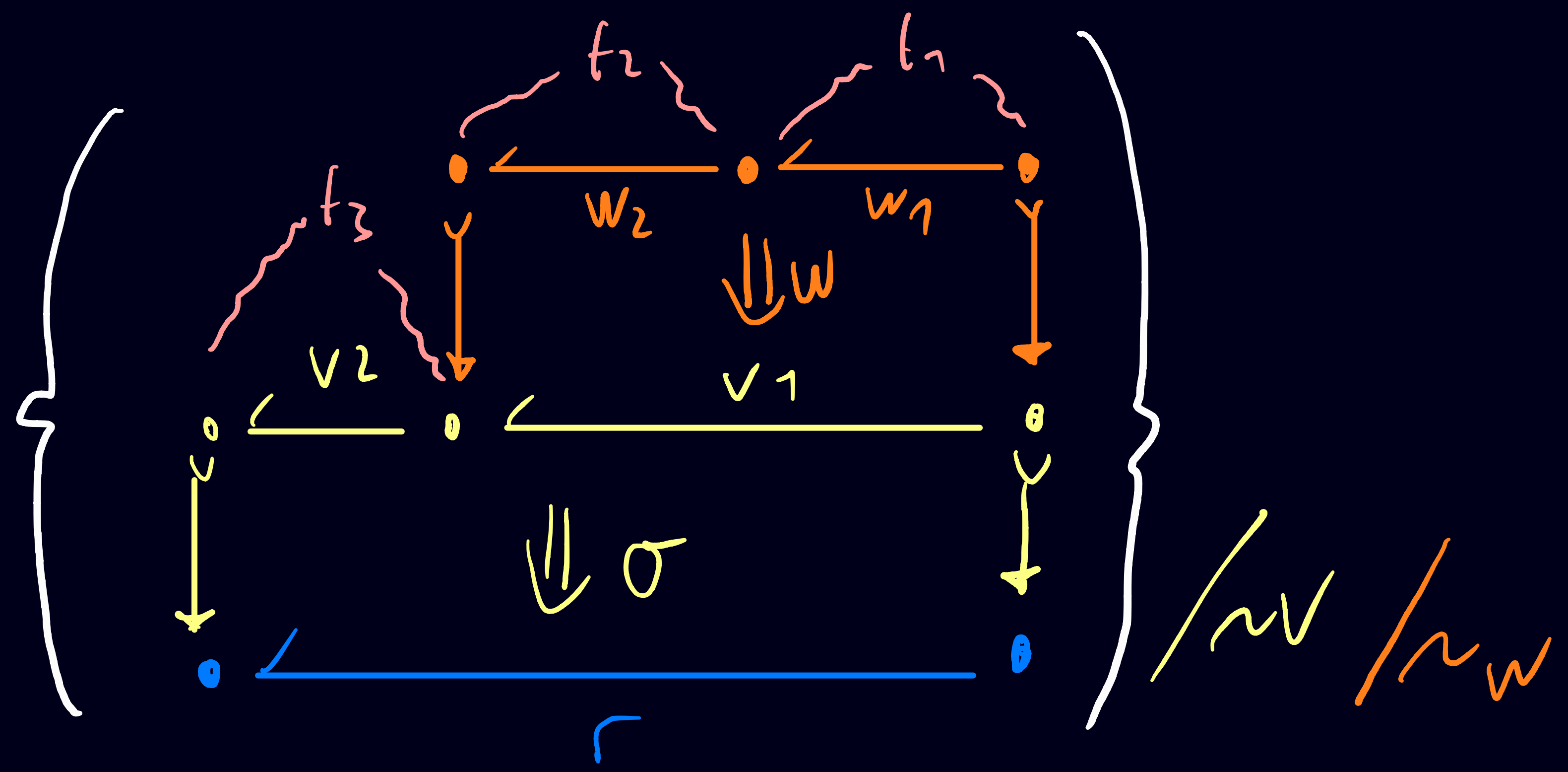
$$F_3 * (F_2 * F_1)(r) = \left\{ \begin{tikzcd} & \bullet \arrow[d] & \bullet \arrow[l, "f_2", "w_2"'] & \bullet \arrow[l, "f_1", "w_1"'] \arrow[d] \arrow[ll, phantom, "\Downarrow w", shift right=6] \\ \bullet \arrow[d] & \bullet \arrow[l, "f_3", "v_2"'] & & \bullet \arrow[ll, "v_1"'] \arrow[d] \\ \bullet & & & \bullet \arrow[lll, "r"] \arrow[uulll, phantom, "\Downarrow \sigma"] \end{tikzcd} \right\} \Big/ \sim_v \Big/ \sim_w$$

Key Result: Weak Associativity of $*$

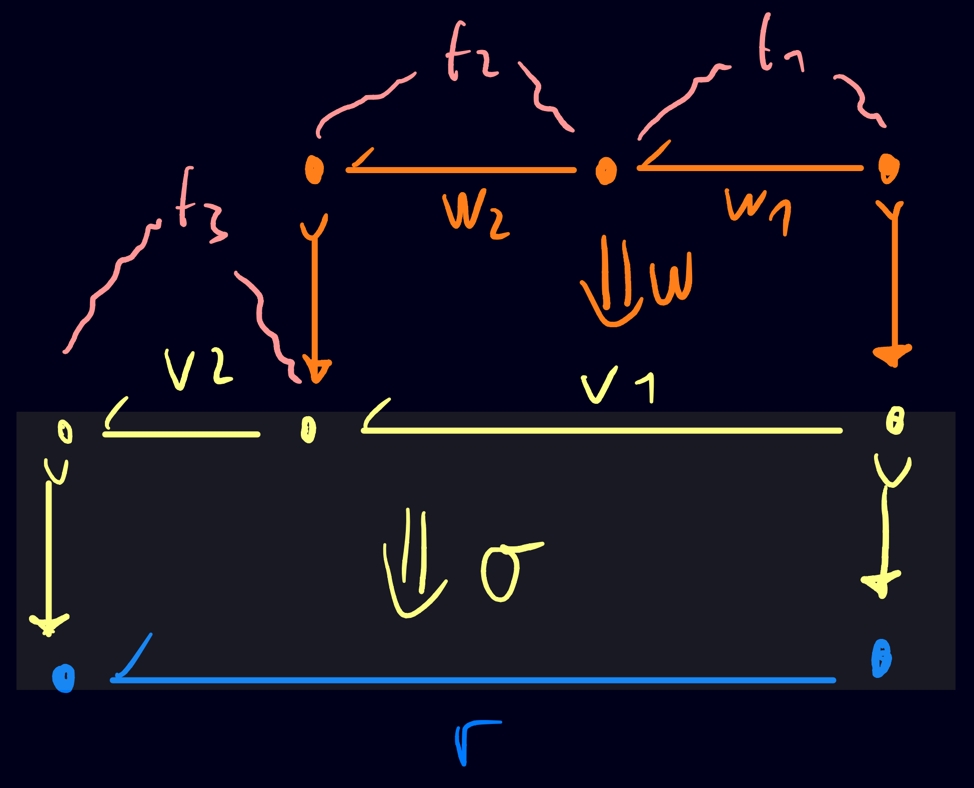
$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{Set}\,,\quad r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

Proof (sketch):

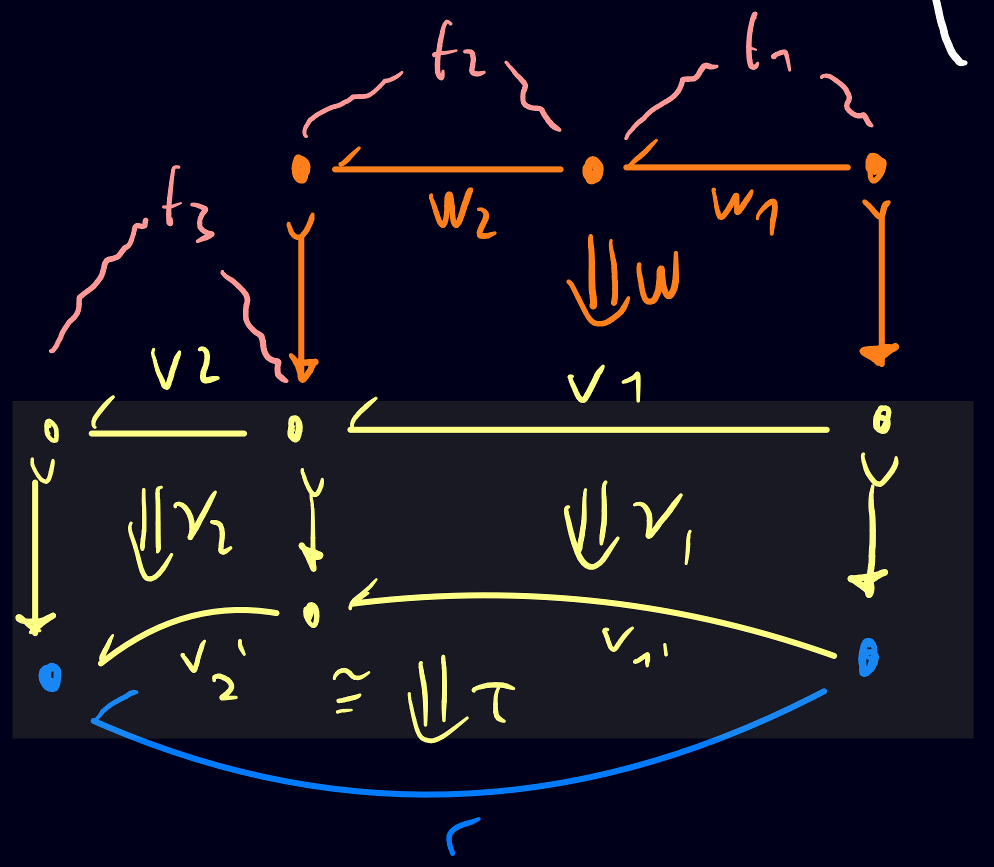
$F_3 * (F_2 * F_1)(r) =$



$\forall$



$: \exists$

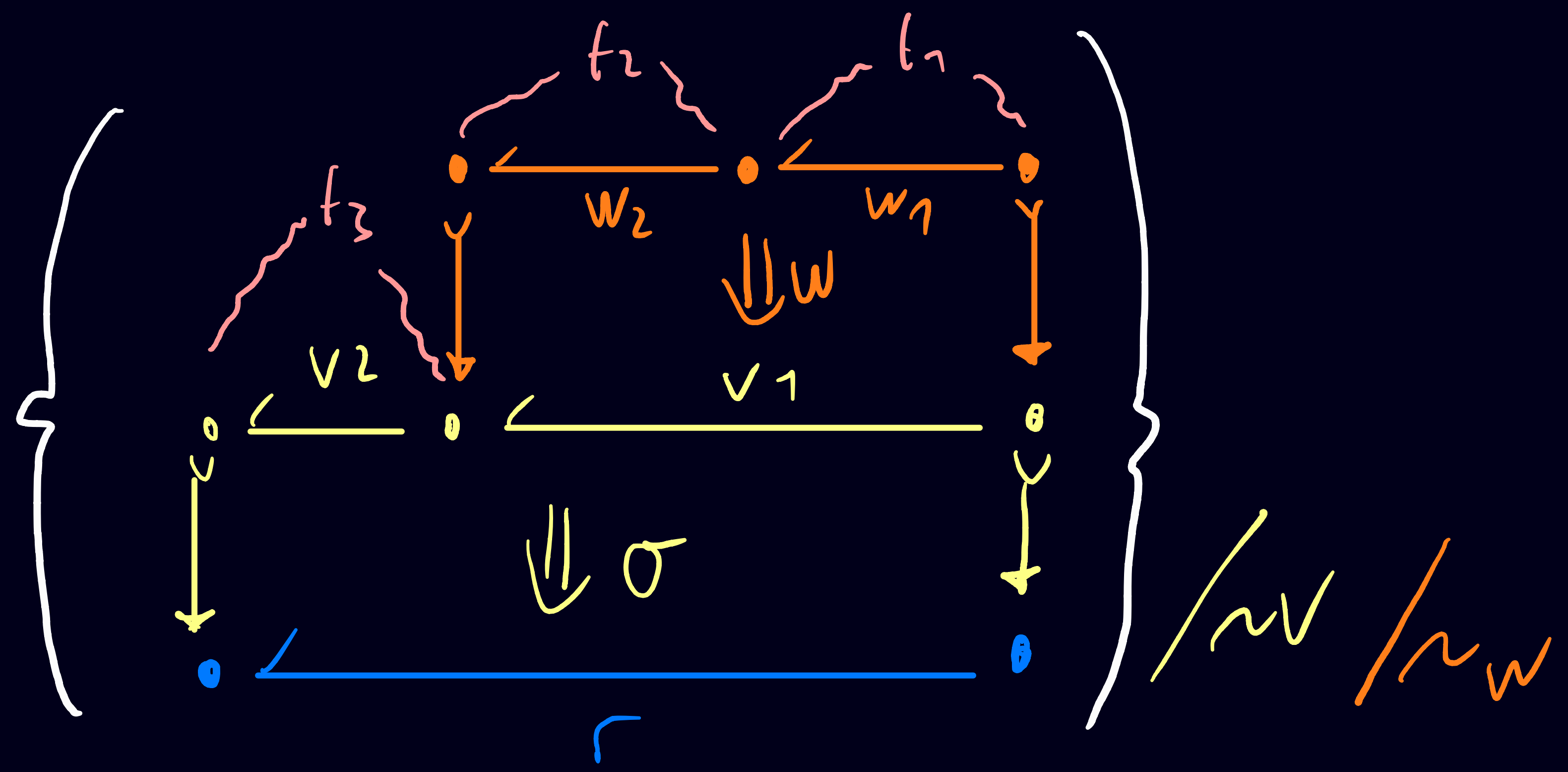


Key Result: Weak Associativity of $*$

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}},\ r \in \mathbb{D}_1 :\ F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

Proof (sketch):

$$F_3 * (F_2 * F_1)(r) = $$

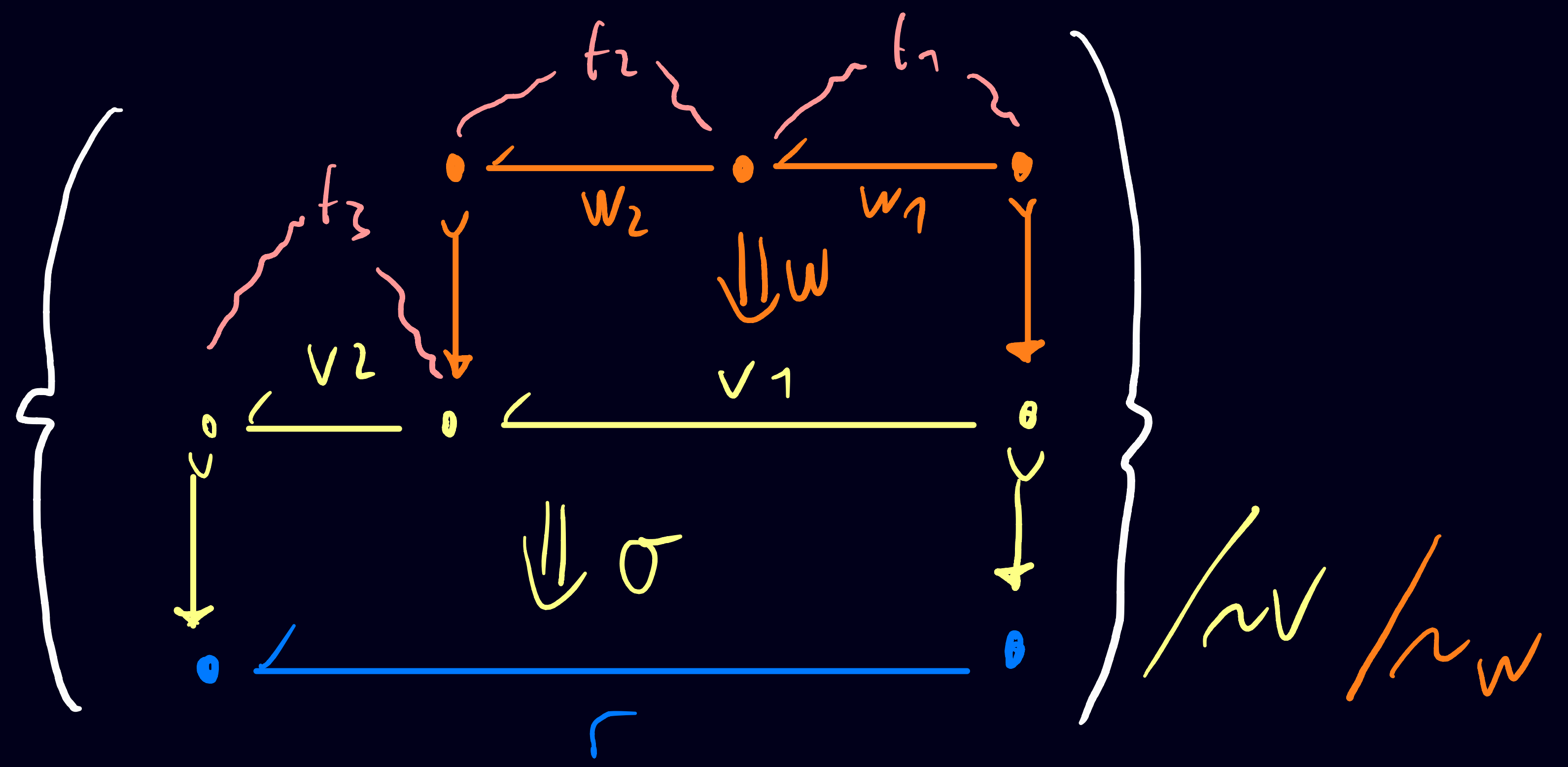


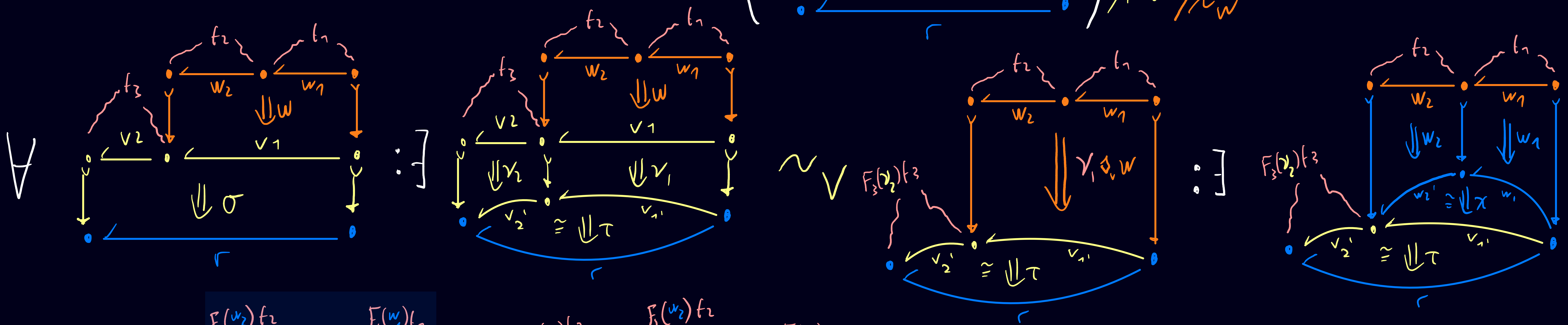
KEY RESULT: WEAK ASSOCIATIVITY OF $*$

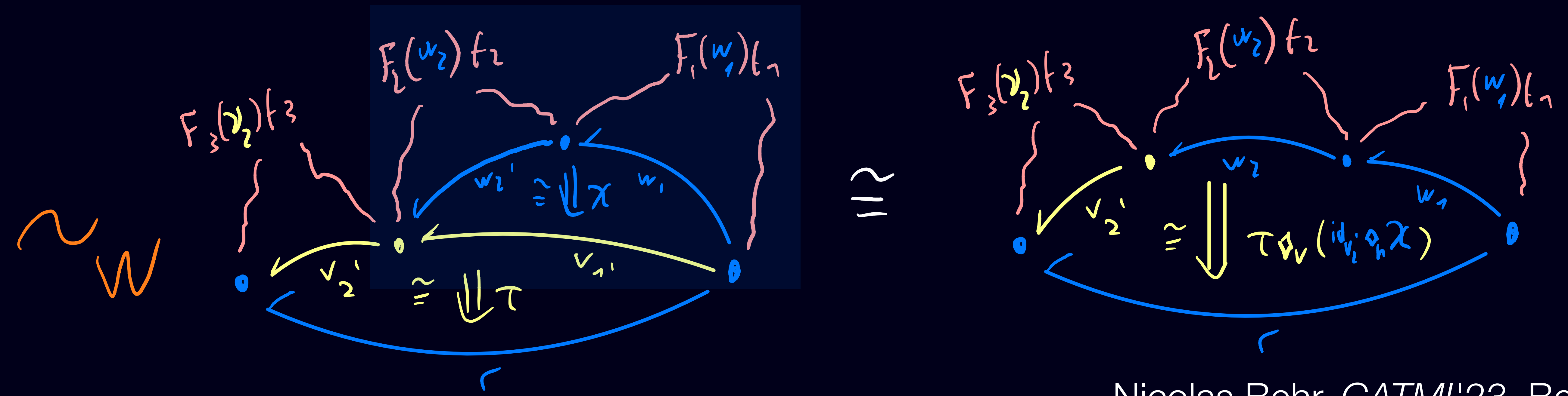
$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}},\quad r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

PROOF (SKETCH):

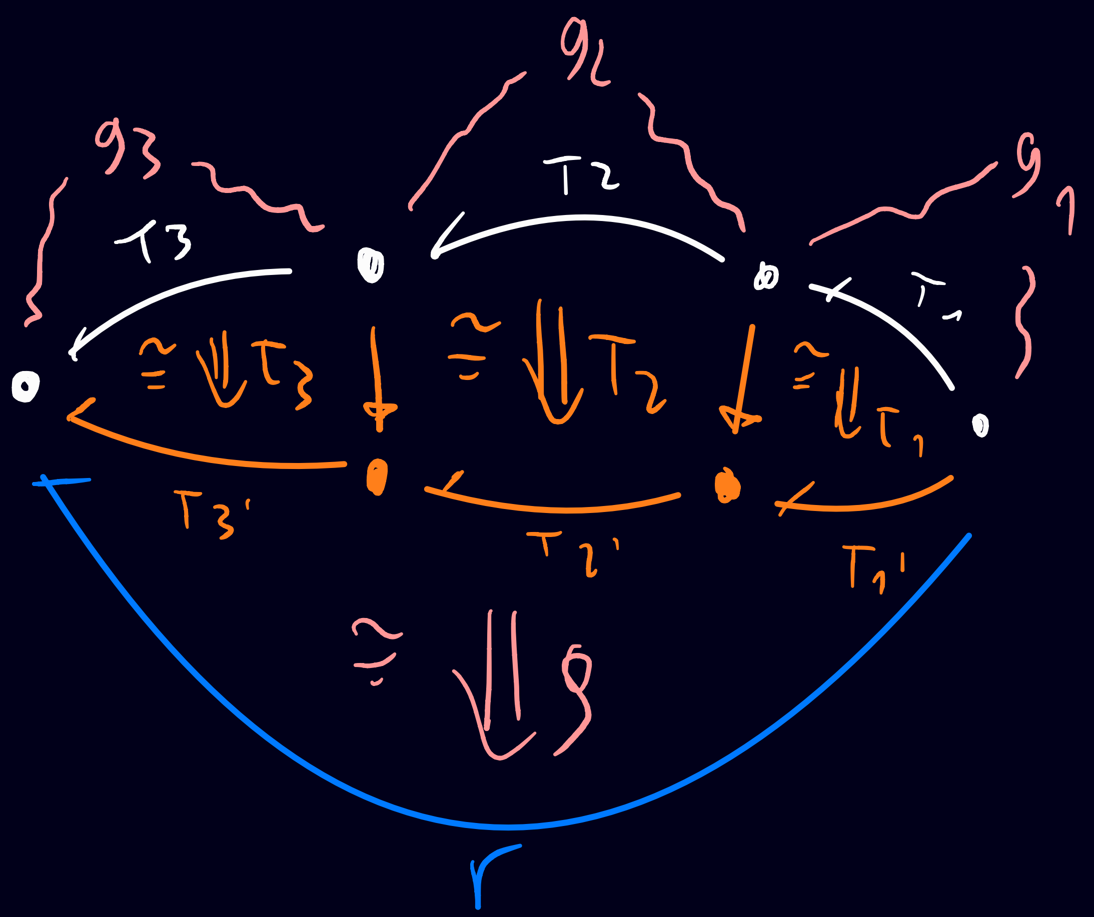
$$F_3 * (F_2 * F_1)(r) =$$



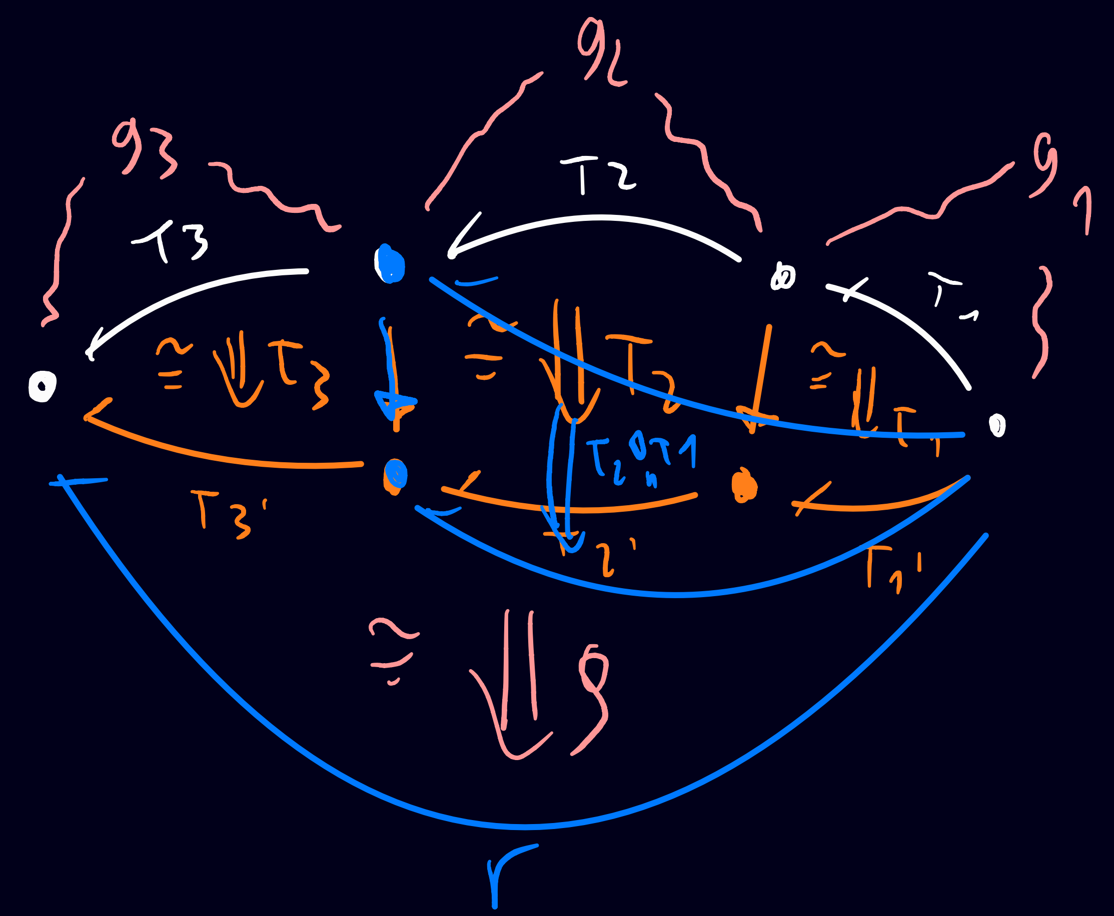




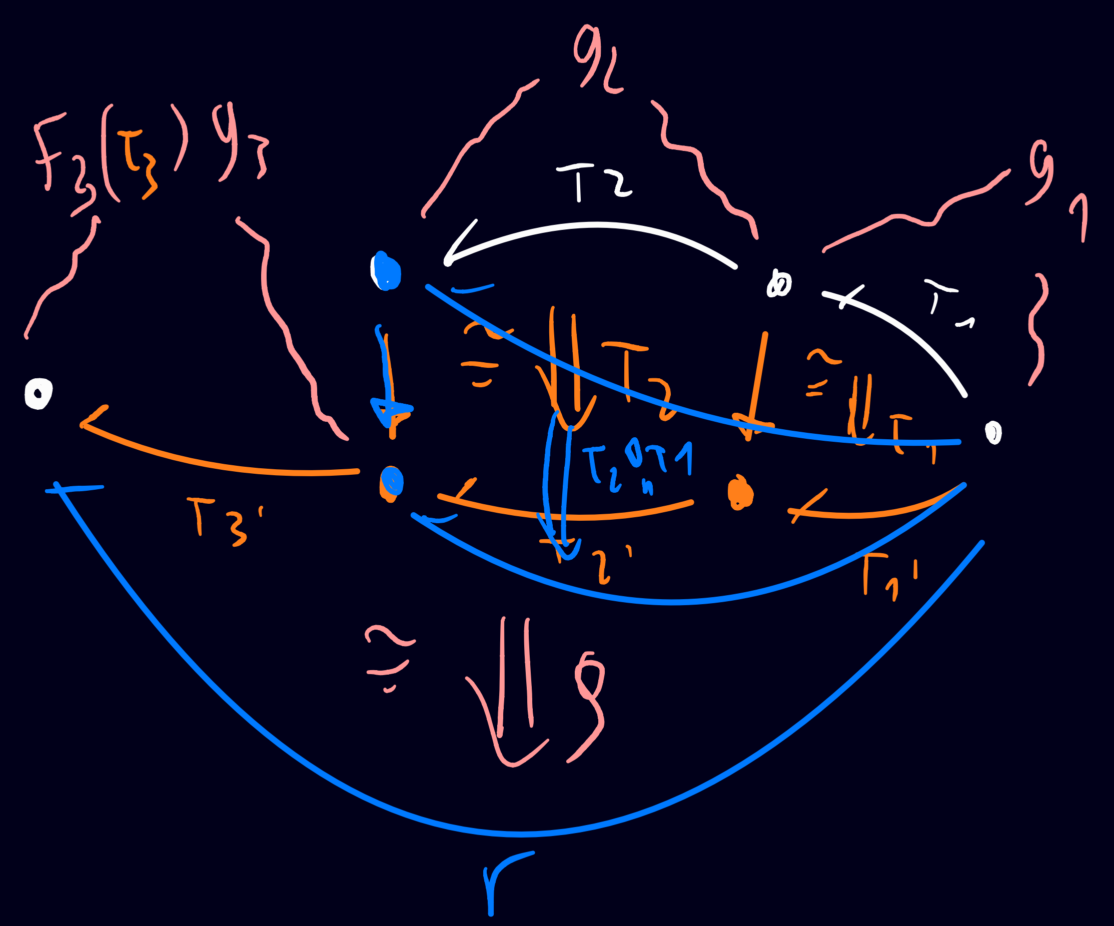
$\forall$



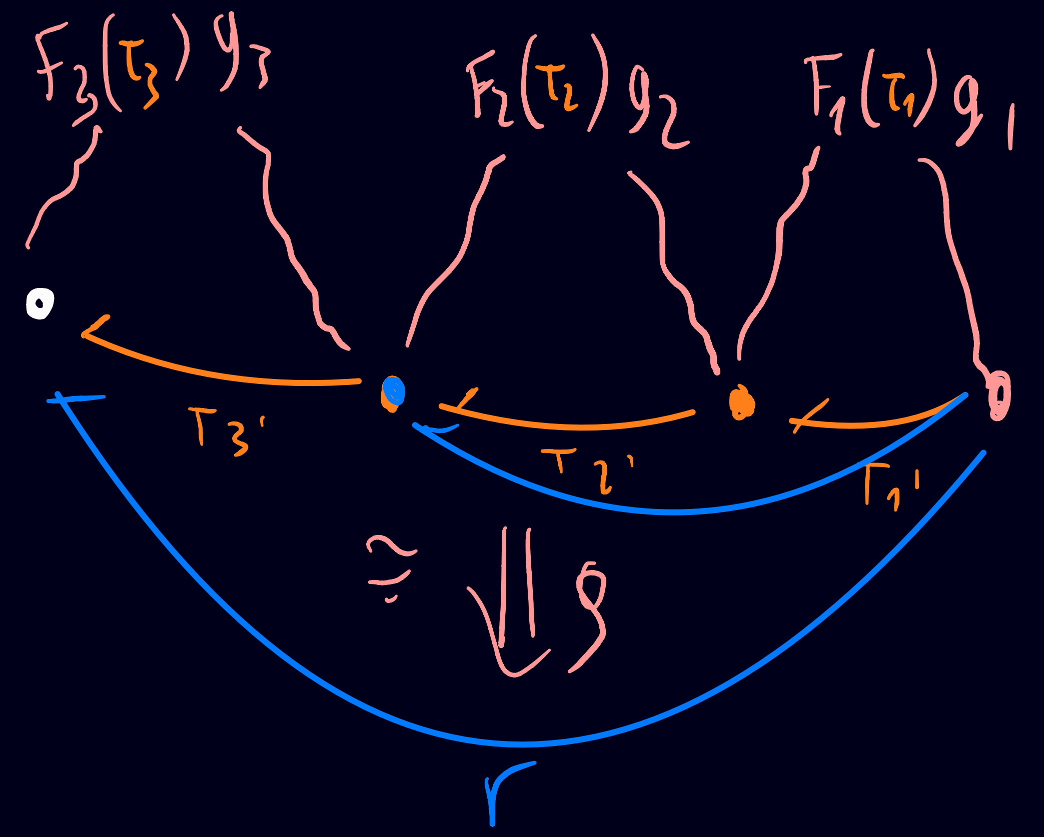
$: \exists$



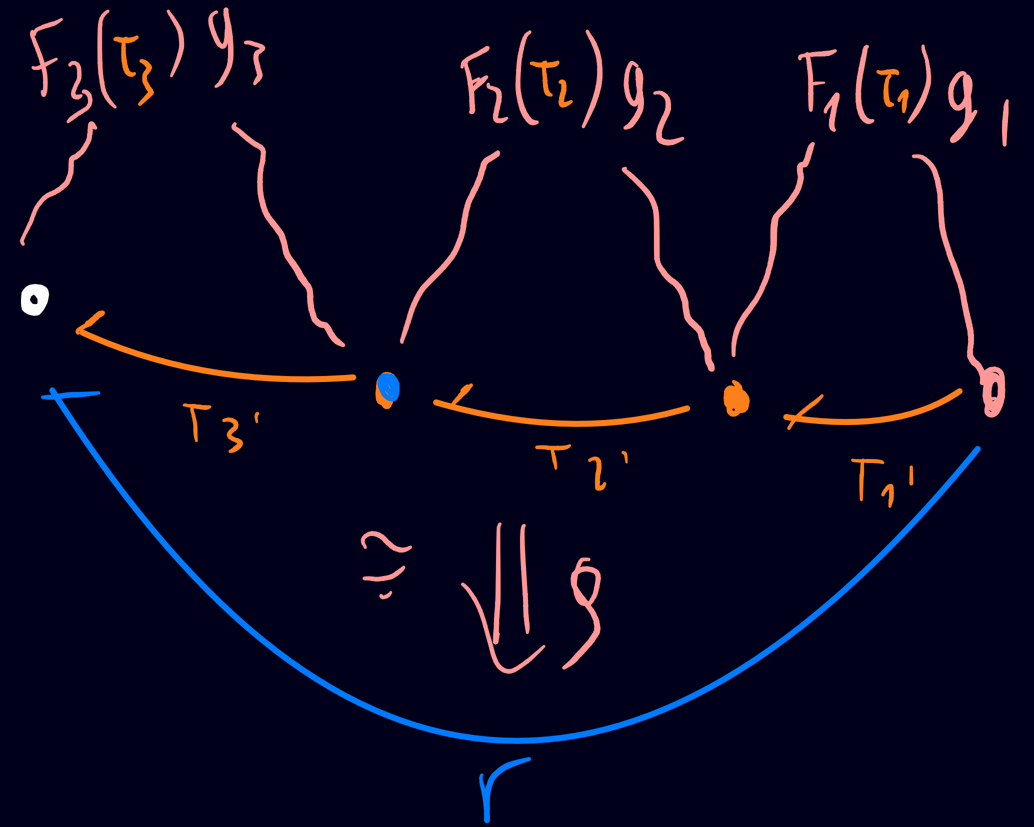
$\sim \checkmark$



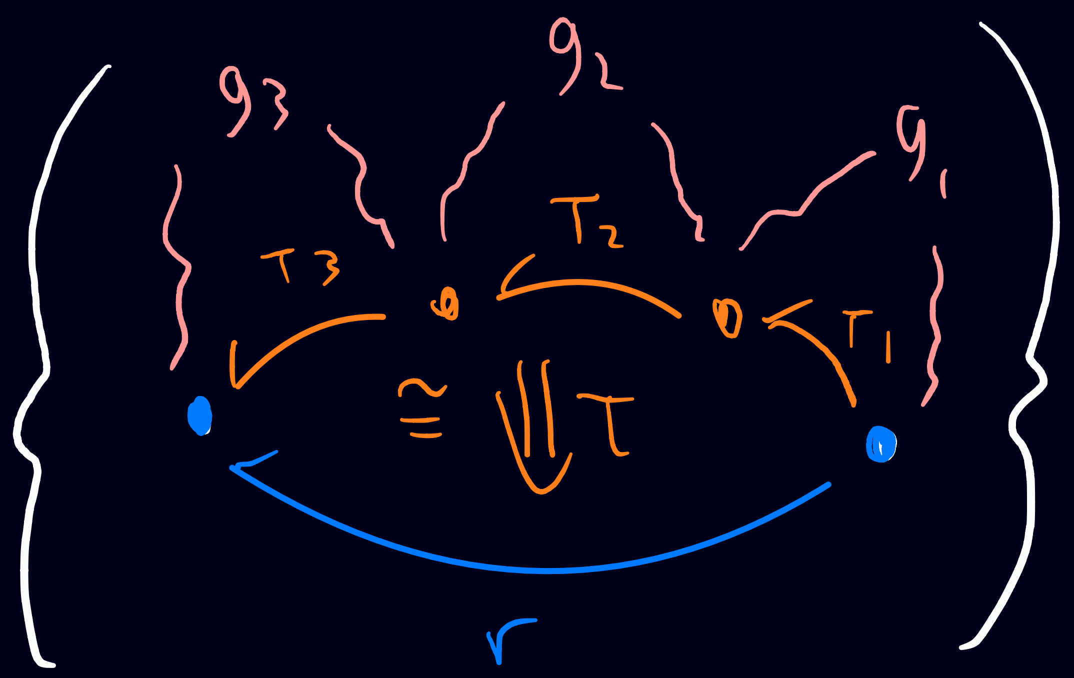
$\sim \checkmark$



$\cong$



$\hookrightarrow \; F_3 * (F_2 * F_1)(r) \cong$



$/\cong_g \; = (F_3 * F_2 * F_3)(r)$

$\square$

$\mathbb{D}_0$ HAS MULTI-SUMS:

$$\forall\ (A,B)\in\mathbb{D}_0\times\mathbb{D}_0 :\ \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j\in\mathfrak{J}_{(A,B)}} :\ \forall \begin{tikzcd} A \arrow[dr, bend right, "x_A"'] & & B \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd} :\ \exists j\in\mathfrak{J}_{(A,B)} :\ \begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, bend right, "x_A"'] & M_j \arrow[d, dashed, "\exists!\ m_j"] & B \arrow[l, "b_j"'] \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$$\wedge\ \forall \begin{tikzcd} A \arrow[dr, "y_A"] \arrow[ddr, bend right, "x_A"'] & & B \arrow[dl, "y_B"'] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd} :\ \exists!\ M_j \xrightarrow{m} Y :\ \begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, "y_A"'] \arrow[ddr, bend right, "x_A"'] & M_j \arrow[d, "m"] & B \arrow[l, "b_j"'] \arrow[dl, "y_B"] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall (A,B)\in\mathbb{D}_0\times\mathbb{D}_0 :\ ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{\mathfrak{J}^{\circ}_{(A,B)}}$$

$\mathbb{D}_0$ HAS MULTI-SUMS:

$$\forall\ (A,B)\in \mathbb{D}_0\times\mathbb{D}_0 : \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j\in J_{(A,B)}} : \forall \begin{tikzcd} A \arrow[dr, bend right, "x_A"'] & & B \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd} : \exists j \in J_{(A,B)} : \begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, bend right, "x_A"'] & M_j \arrow[d, dotted, "\exists!\ m_j"] & B \arrow[l, "b_j"'] \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$$\wedge\ \forall \begin{tikzcd} A \arrow[dr, "y_A"] \arrow[ddr, bend right, "x_A"'] & & B \arrow[dl, "y_B"'] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd} : \exists!\ M_j \xrightarrow{m} Y : \begin{tikzcd} A \arrow[r, "a_j"] \arrow[ddr, "y_A"] \arrow[dddr, bend right, "x_A"'] & M_j \arrow[dd, "m"] & B \arrow[l, "b_j"'] \arrow[ddl, "y_B"'] \arrow[dddl, bend left, "x_B"] \\ & & \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall (A,B)\in\mathbb{D}_0\times\mathbb{D}_0 : ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{J^{\circ}_{(A,B)}}$$

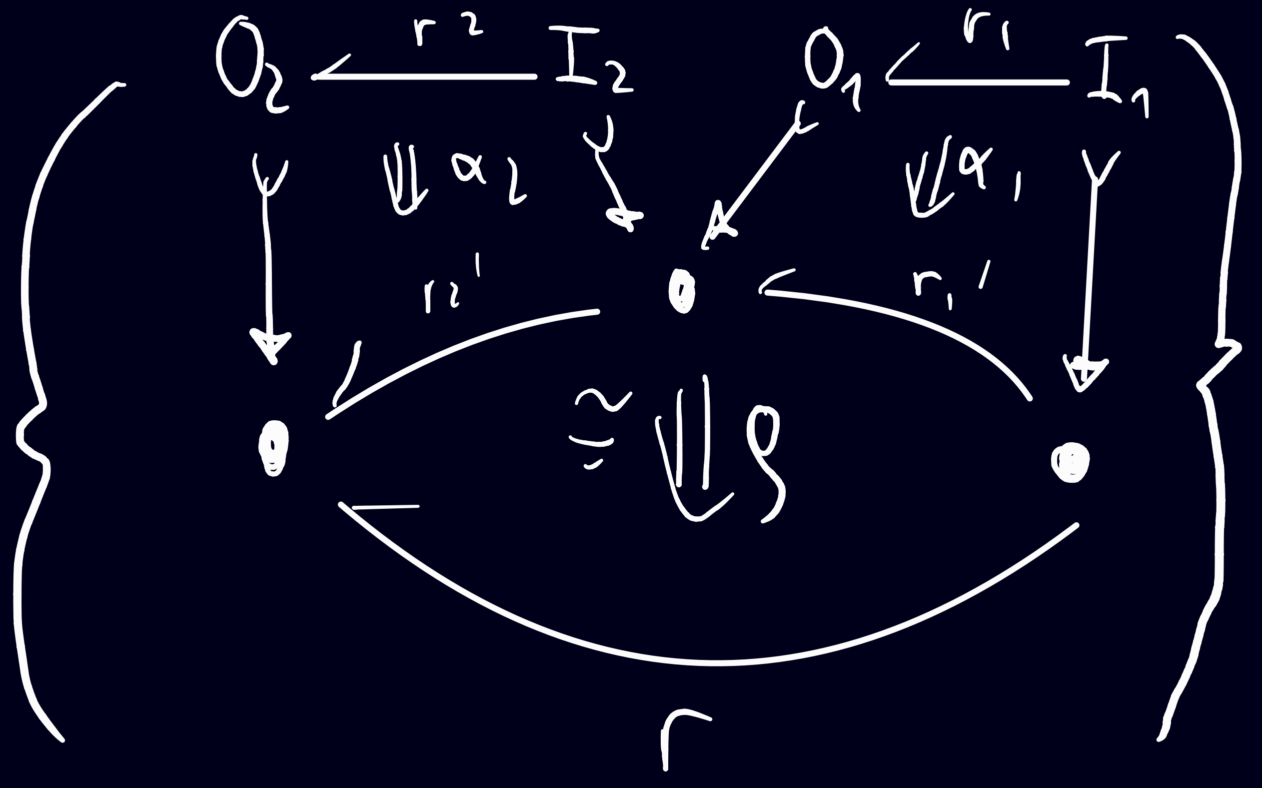
$S: \mathbb{D}_1 \to \mathbb{D}_0$ IS A MULTI-OPFIBRATION

$$\forall \begin{tikzcd} \bullet \arrow[dd, bend right, "n''"'] & & \bullet \arrow[ll, "r"'] \arrow[d, "m"] \arrow[ddll, phantom, "\Downarrow \alpha''"] \\ & & \bullet \arrow[d, "m'"] \\ \bullet & & \bullet \arrow[ll, "r''"] \end{tikzcd} : \exists \begin{tikzcd} \bullet \arrow[d, "n"'] & \bullet \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow \alpha"] \\ \bullet \arrow[d, "n'"'] & \bullet \arrow[l, "r'"'] \arrow[d, "m'"] \arrow[dl, phantom, "\Downarrow \alpha'"] \\ \bullet & \bullet \arrow[l, "r''"] \end{tikzcd} : n' \circ n = n'' \ \wedge\ \alpha' \circ_v \alpha = \alpha''$$

$T: \mathbb{D}_1 \to \mathbb{D}_0$ IS A RESIDUAL MULTI-OPFIBRATION

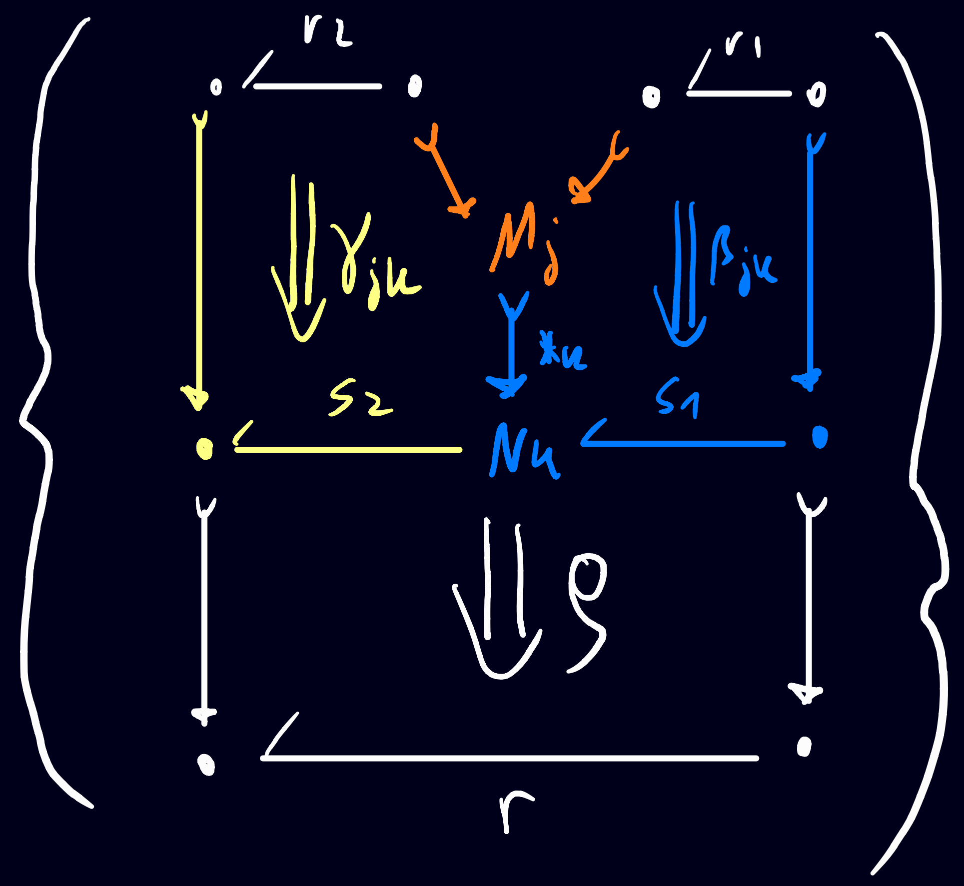
$$\forall \begin{tikzcd} \bullet \arrow[d, "n"'] \arrow[dd, bend right, "n''"'] & & \bullet \arrow[ll, "r"'] \arrow[dd, bend left, "m''"] \arrow[ddll, phantom, "\Downarrow \alpha''"] \\ \bullet \arrow[d, "n'"'] & & \\ \bullet & & \bullet \arrow[ll, "r''"] \end{tikzcd} : \exists \begin{tikzcd} \bullet \arrow[d, "n"'] & & \bullet \arrow[ll, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow \beta"] \\ \bullet \arrow[d, "n'"'] \arrow[r, "n_x"] & \bullet \arrow[dl, "\bar{n}"] & \bullet \arrow[l, "r_x"'] \arrow[d, "m'"] \arrow[dl, phantom, "\Downarrow \beta'"] \\ \bullet & & \bullet \arrow[ll, "r''"] \end{tikzcd} : \bar{n} \circ n_x = n' \ \wedge\ m' \circ m = m'' \ \wedge\ \beta' \circ_v \beta = \beta''$$

17 KEY RESULT: CATEGORIFICATION OF THE RULE ALGEBRA

CLAIM: $(\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$ 

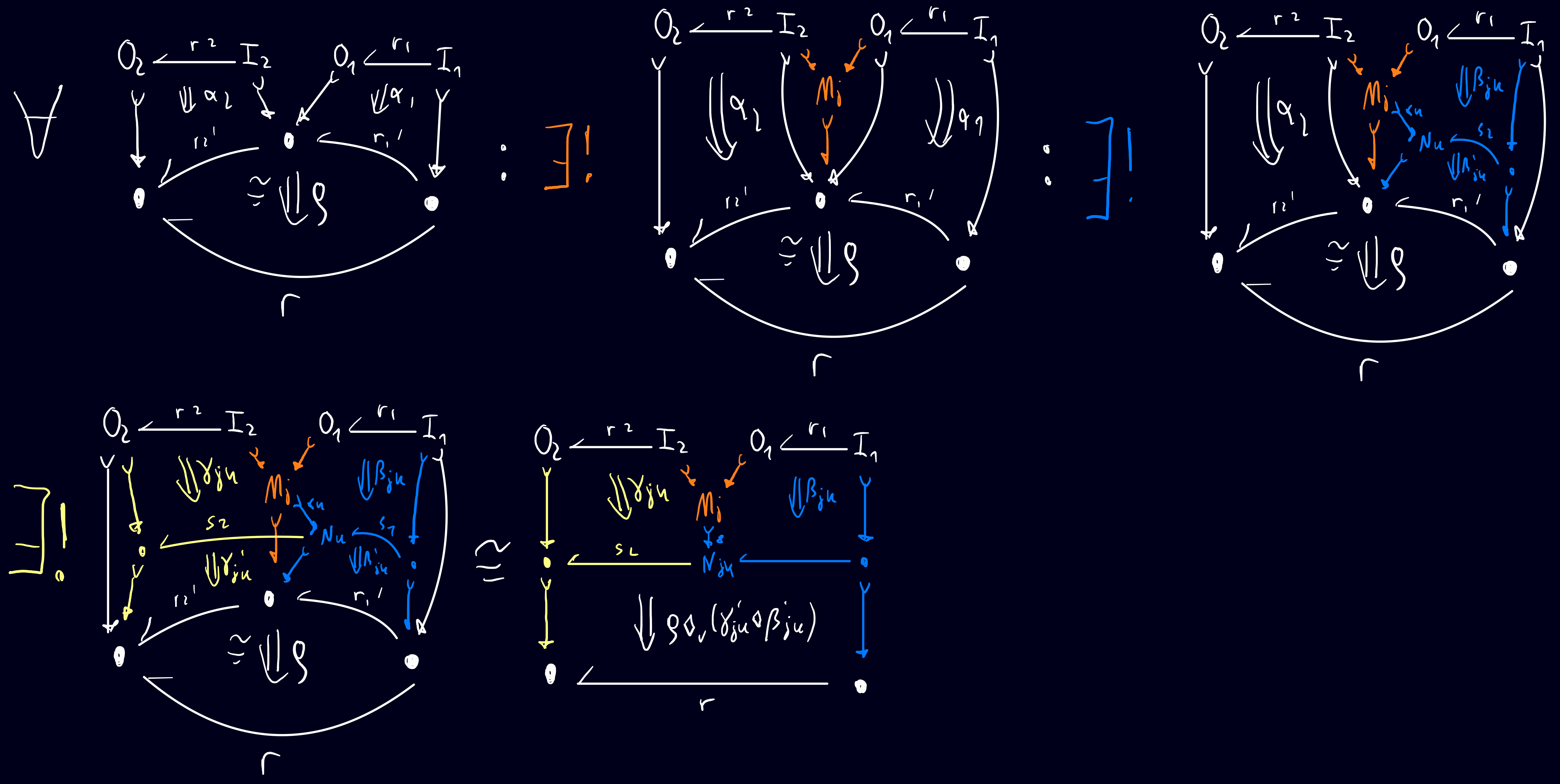
$/\cong_y$

$$\cong \coprod_{\substack{(O_2 \rightarrowtail M_j \leftarrowtail I_1) \\ \in ms(O_2, I_1)}} \;\; \coprod_{\beta_{jk} \in T^*(O_1 \rightarrowtail M_j)} \;\; \coprod_{\gamma_{jk} \in S^*(I_2 \rightarrowtail M_j \xrightarrow{*_k} N_k)}$$

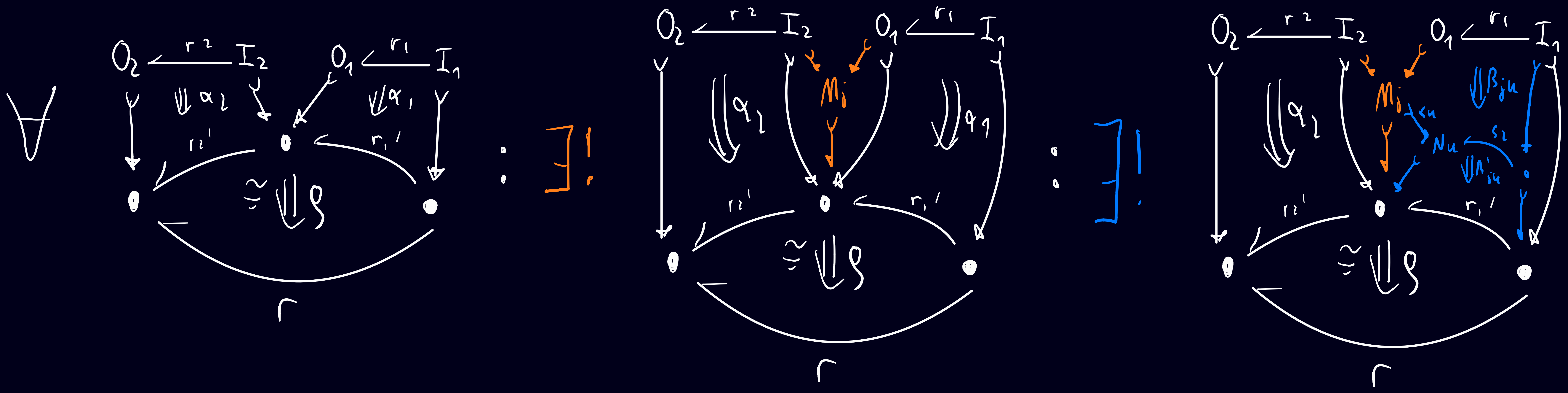


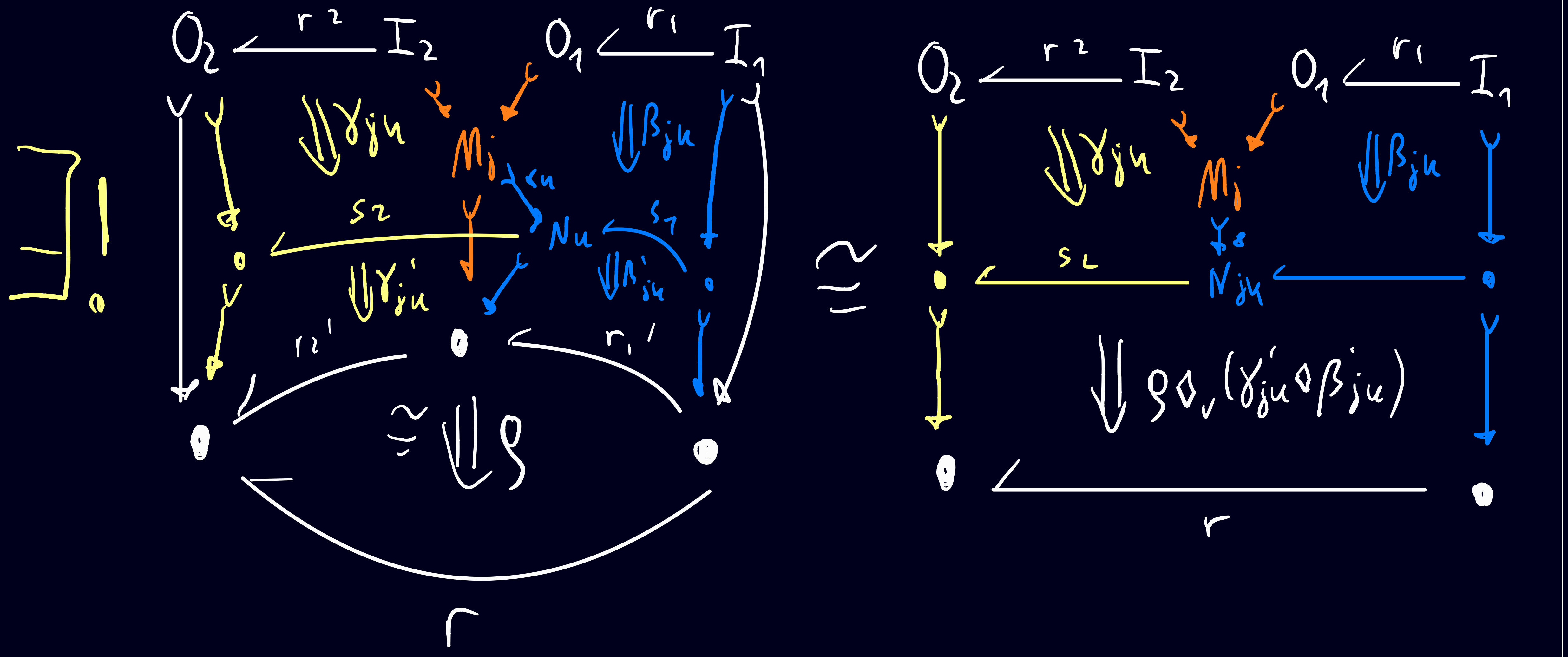
$$=: \hat{\Delta}_{r_2 \odot r_1}(r)$$

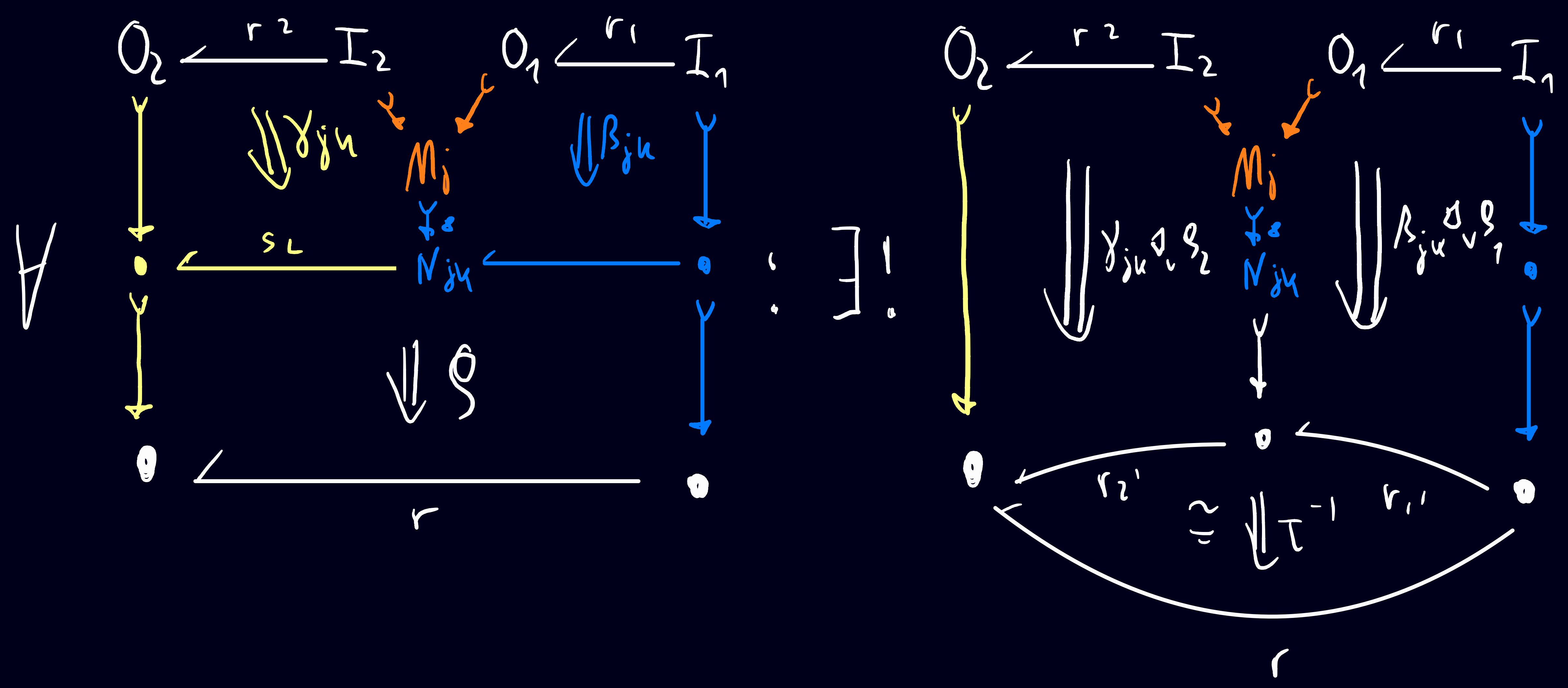
17 Proof (sketch): Assuming chosen cleavages:



PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:



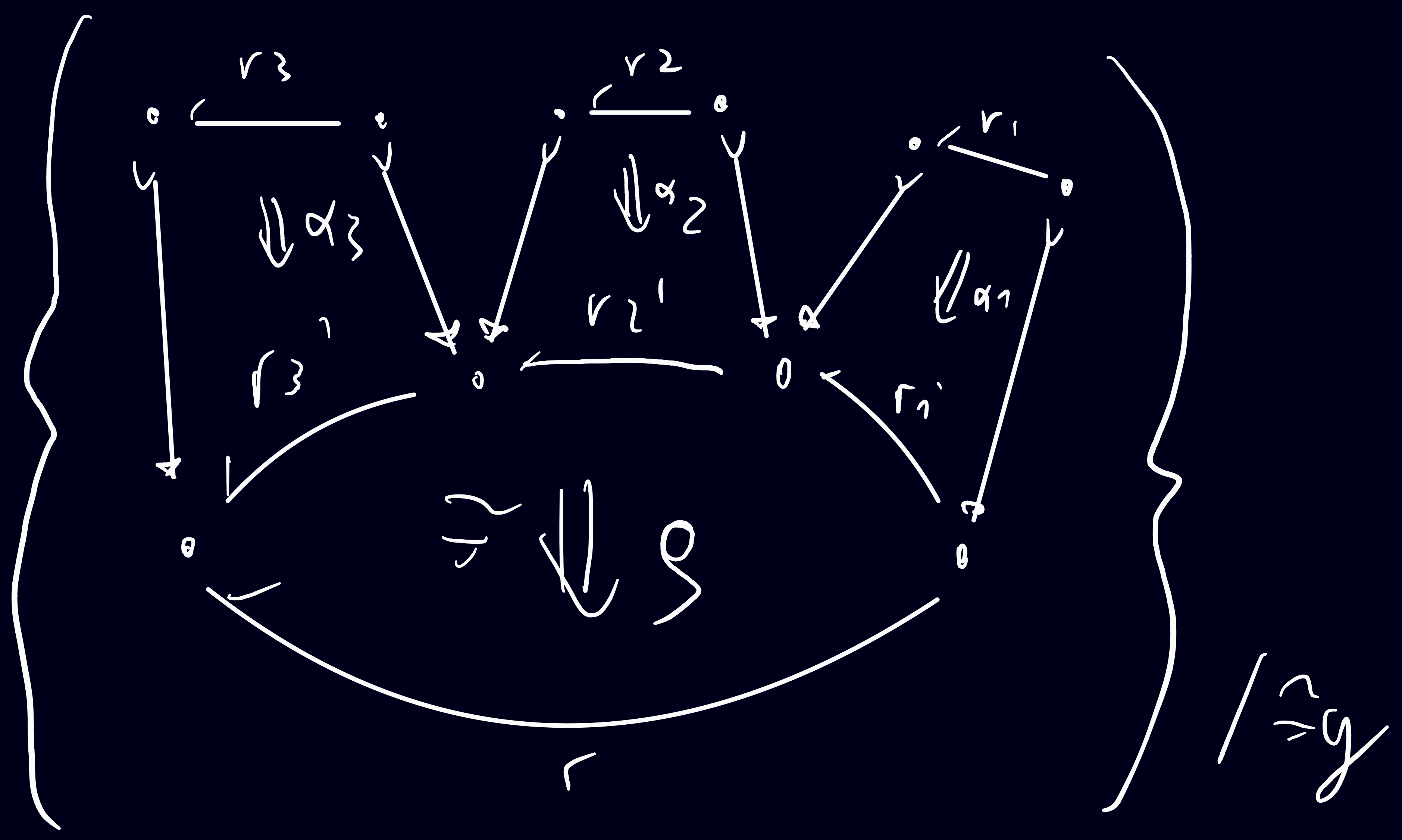




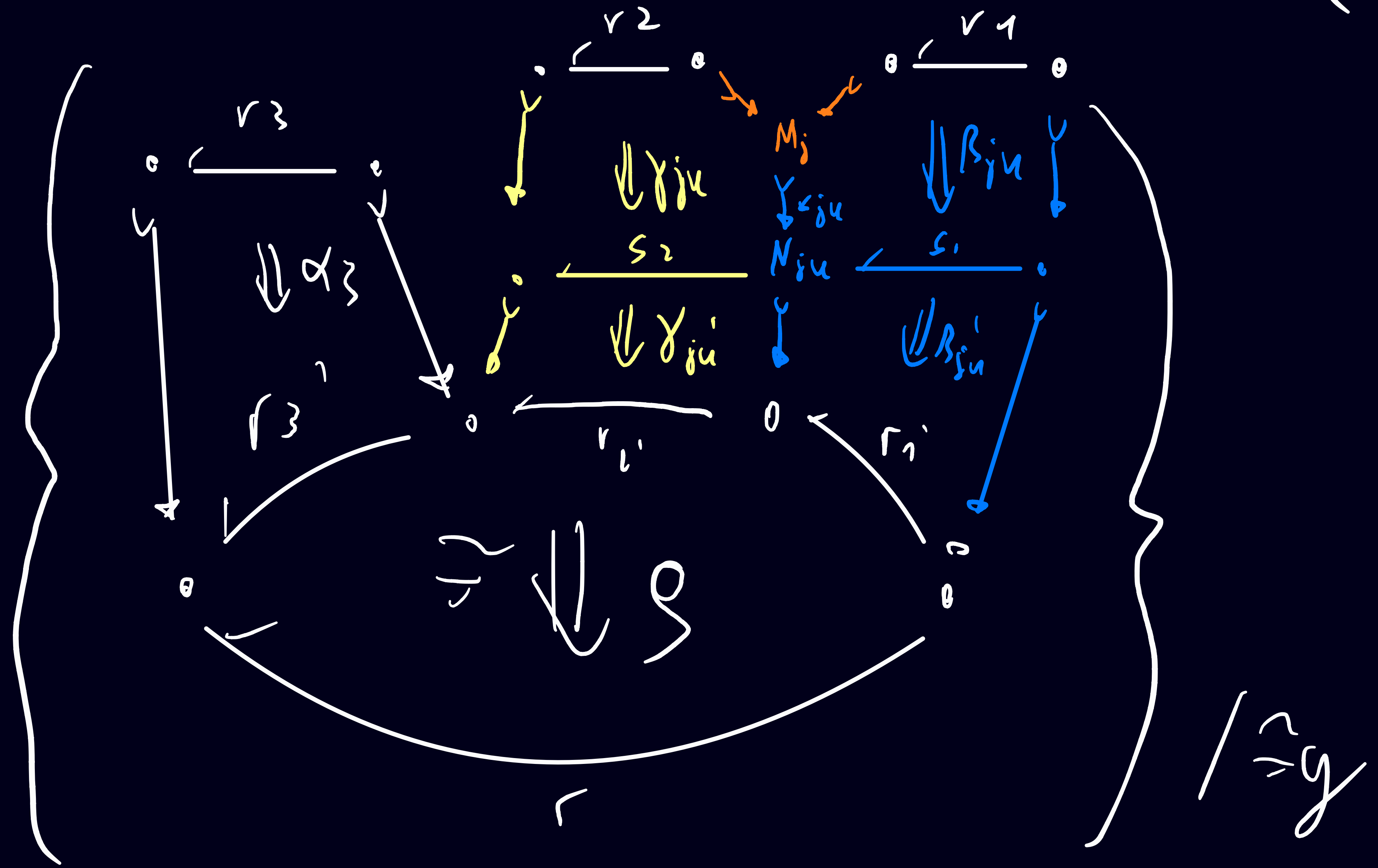
CLAIM: $\hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r) \cong \hat{\Delta}_{(r_3 \odot r_2) \odot r_1}(r)$

PROOF (SKETCH):

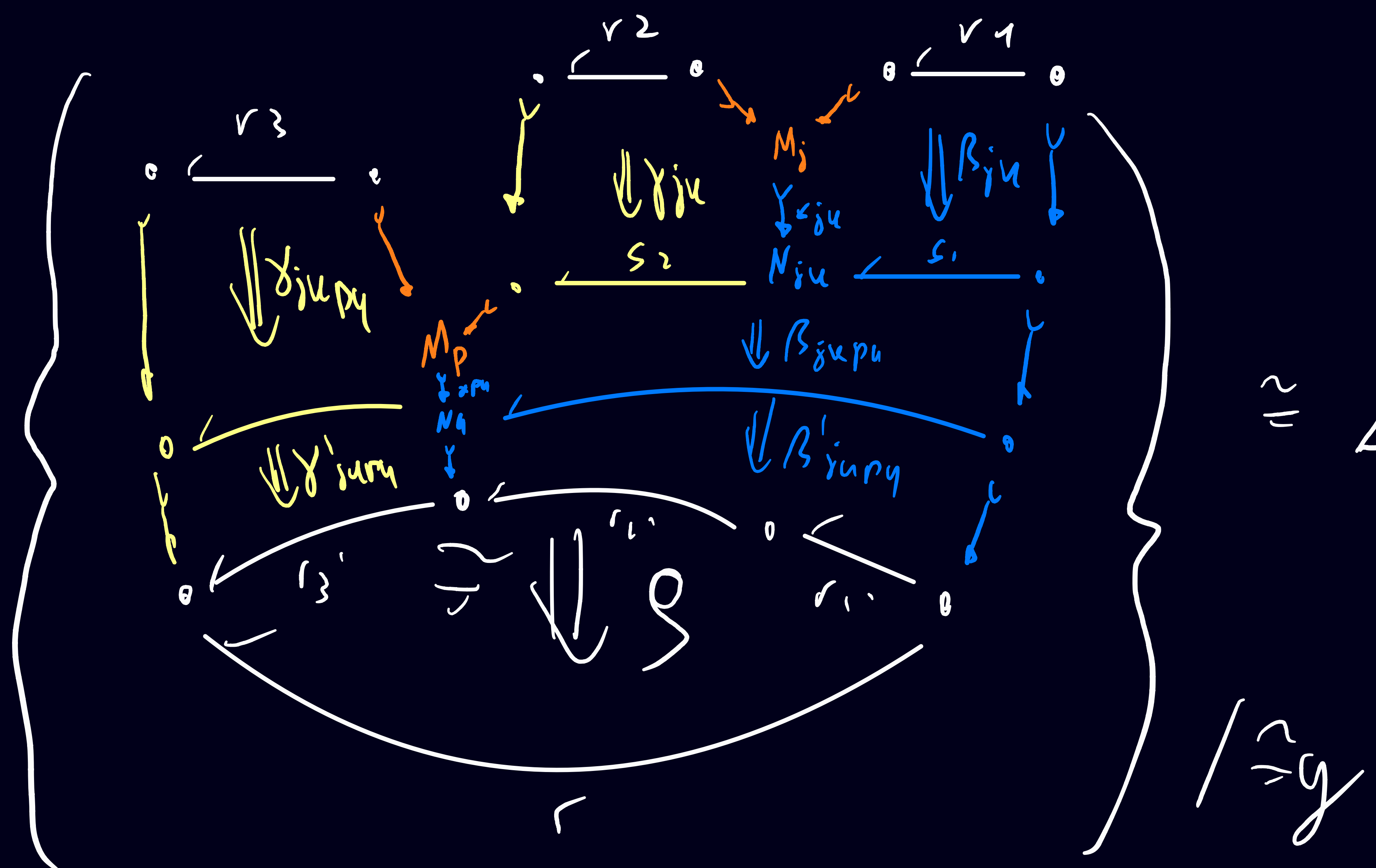
$(\hat{\Delta}_{r_3} * \hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$



$\cong$



$\cong$



$\cong \hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r)$

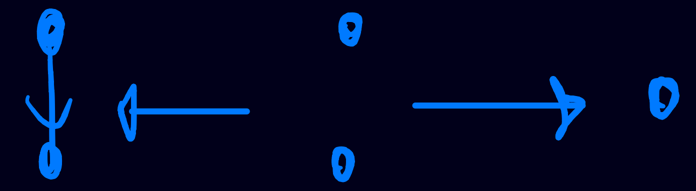
$\square$

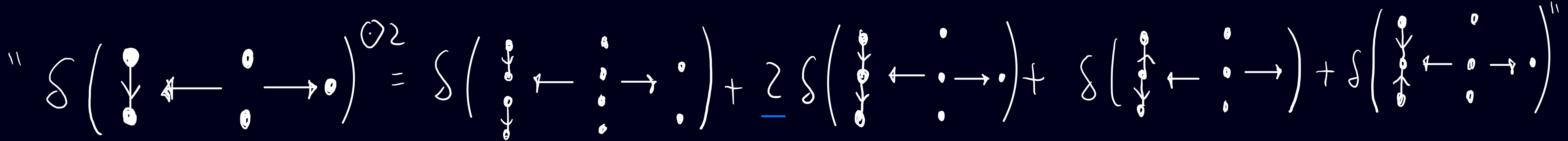
Decategorification of the Rule Algebra:

$$r_2 \odot r_1 := \coprod_{\substack{(O_2 \rightarrowtail M_j \leftarrowtail I_1)\\ \in \mathsf{ms}(O_2, I_1)}} \;\; \coprod_{\beta_{jk} \in T^*(O_1 \rightarrowtail M_j)} \;\; \coprod_{\gamma_{jk} \in S^*(I_2 \rightarrowtail M_j \xrightarrow{*_k} N_k)} \left\{ \begin{tikzcd} \bullet \arrow[dd, tail] & \bullet \arrow[l, "r_2"'] \arrow[dr, tail] & & \bullet \arrow[dl, tail] & \bullet \arrow[l, "r_1"'] \arrow[dd, tail] \\ \arrow[r, phantom, "\Downarrow \gamma_{jk}"] & & M_j \arrow[d, tail, "*_k"] & \arrow[r, phantom, "\Downarrow \beta_{jk}"] & \\ \bullet & & N_k \arrow[ll, "s_2"'] & & \bullet \arrow[ll, "s_1"'] \end{tikzcd} \right\}$$

$$\big\downarrow$$

$$\delta(r_2) *_R \delta(r_1) := \sum_{\ldots} \sum_{\ldots} \sum_{\ldots} \delta\Big(\big[s_2 \diamond_k s_1 \big]_{\cong \mathbb{D}_1} \Big)$$

EXAMPLE: SELF-COMPOSITIONS OF THE REWRITING RULE 

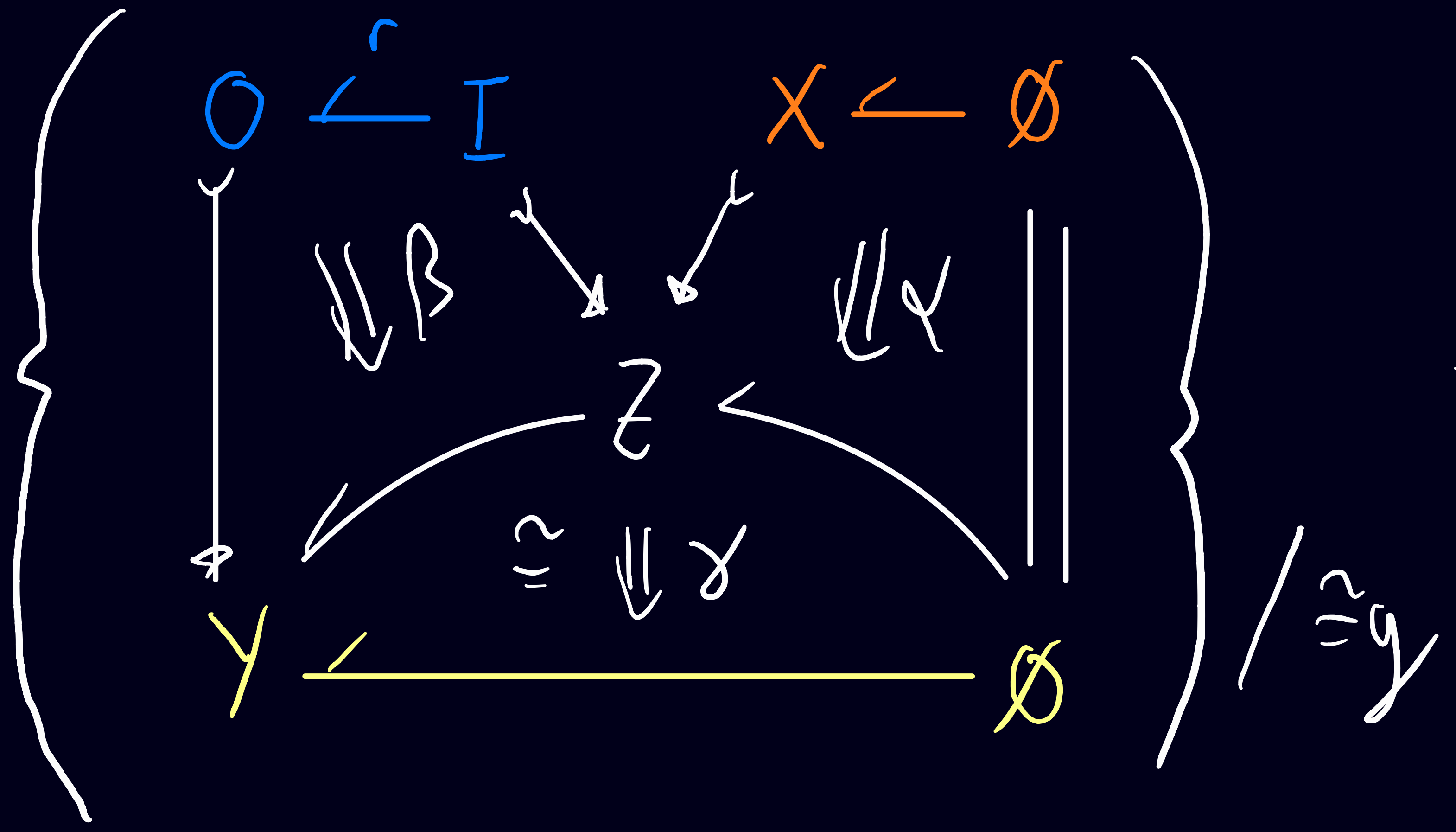


5 CONTRIBUTIONS TO $\hat{\Delta}_{r_2 \circ r_1}$:

COUNTING REWRITING SEQUENCES:

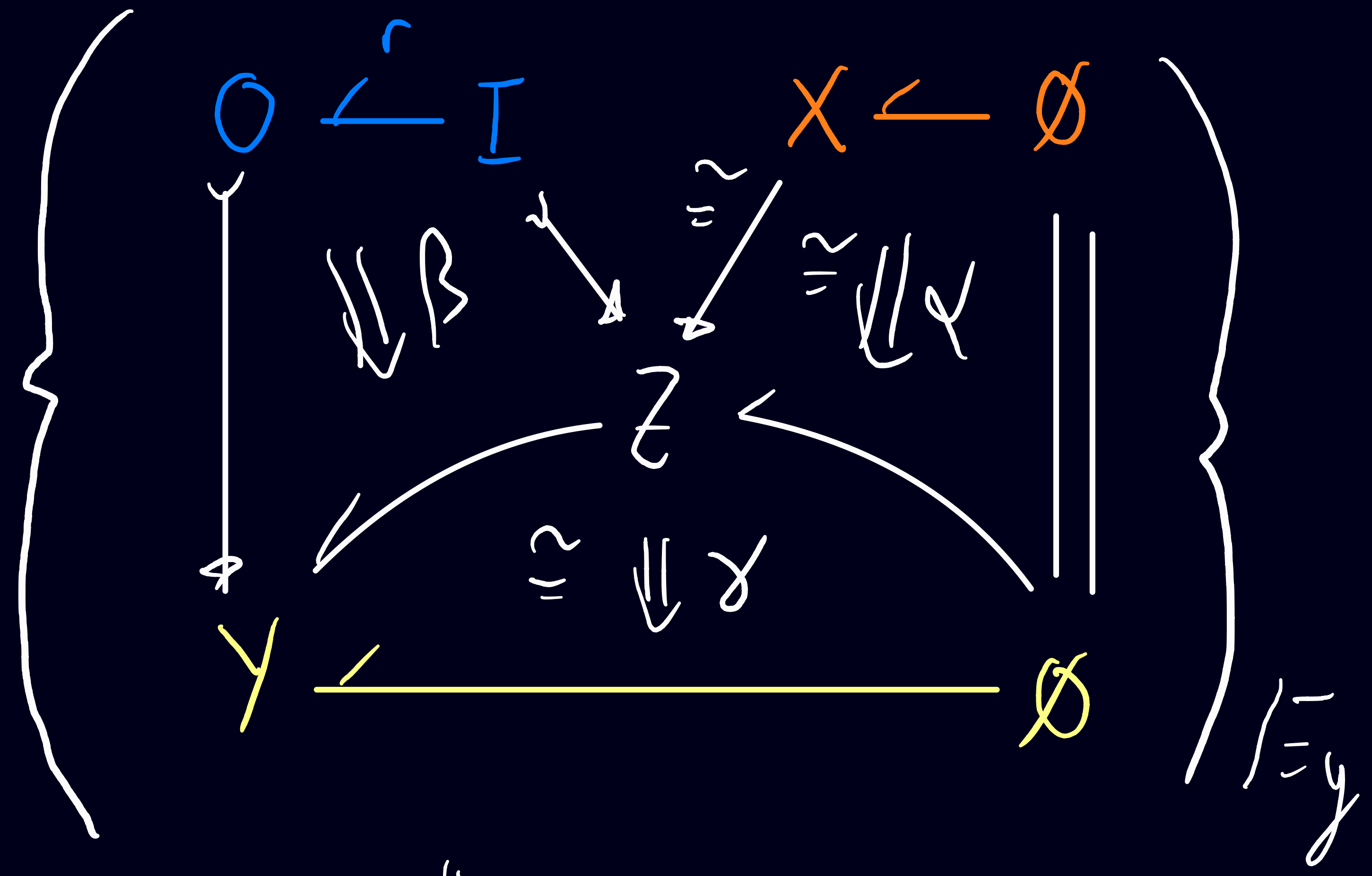
$$\text{“}\ \varrho(\mathbb{S}(r))\,|X\rangle = \varrho(\mathbb{S}(r))\,\varrho(\mathbb{S}(X \leftarrow \emptyset))\,|\emptyset\rangle = \sum_{\alpha} \varrho(\mathbb{S}(r_{\alpha}(X) \leftarrow \emptyset))\,|\emptyset\rangle\ \text{”}$$

$$(\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \cong$$

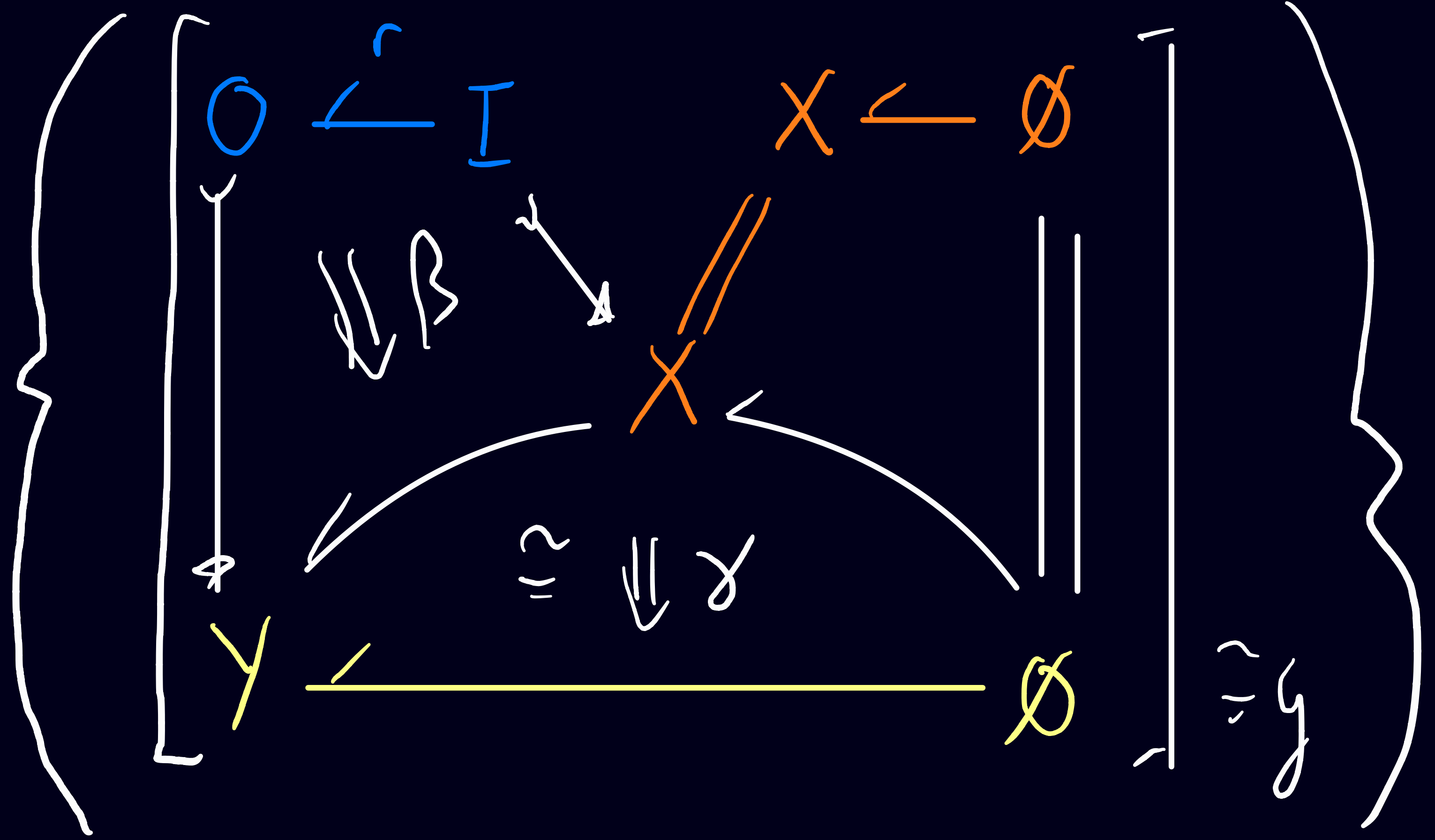


$\cong$

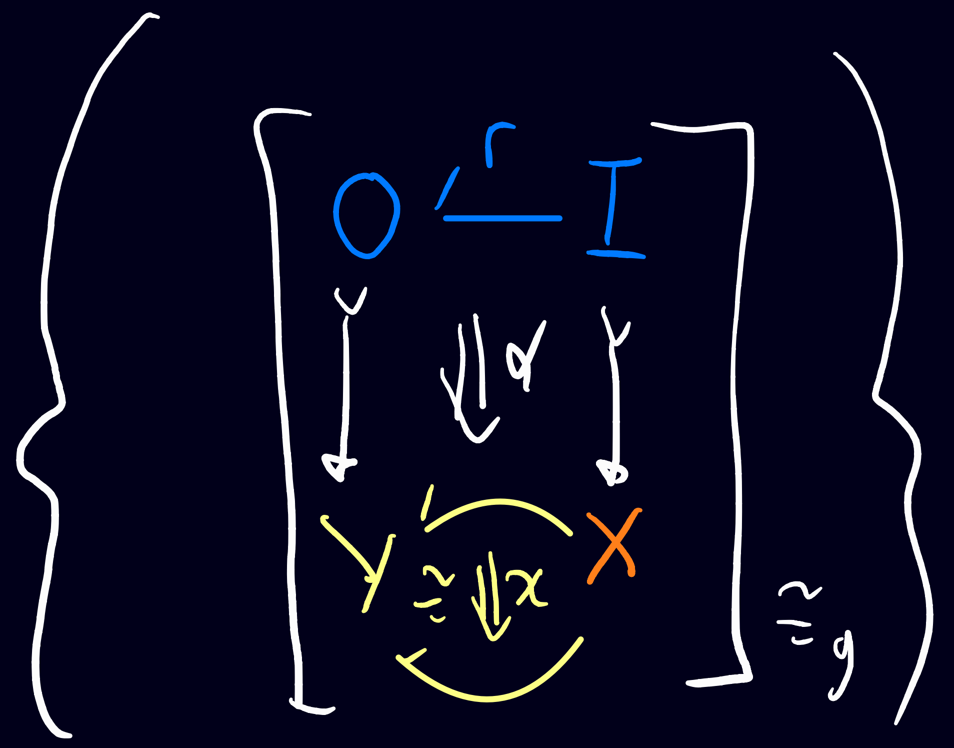
$\mathbb{S}$ is a MOPF $\Rightarrow$ "lifts" isos!



$\cong$



$\cong$



✓

COUNTING REWRITING SEQUENCES:

$$\text{"}\ \varrho(\delta(r))|X\rangle = \varrho(\delta(r))\,\varrho(\delta(X \leftarrow \emptyset))|\emptyset\rangle = \sum_{\alpha} \varrho(\delta(r_\alpha(X) \leftarrow \emptyset))|\emptyset\rangle\ \text{"}$$

$$(\hat{\Delta}_r * \hat{\Delta}_{X \leftarrow \emptyset})(Y \leftarrow \emptyset) \cong \left\{ \left[\begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y \arrow[r, bend left] \arrow[r, phantom, "\cong\Downarrow x"] & X \arrow[l, bend left] \end{tikzcd} \right]_{\cong_g} \right\}$$

↳ DECATEGORIFICATION:

$$\varrho(\delta(r))_{YX} := \left| \left\{ \left[\begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ Y \arrow[r, bend left] \arrow[r, phantom, "\cong\Downarrow x"] & X \arrow[l, bend left] \end{tikzcd} \right]_{\cong_g} \right\} \right| \in \mathbb{Z}_{\geq 0}$$

"# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0}(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y\leftarrow\emptyset)\cong\coprod_{\alpha\in\mathbb{D}_1}\left\{\begin{tikzcd} O \arrow[d] & I \arrow[l, "r"'] \arrow[d] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd}\right\}\Big/\sim_Y$$

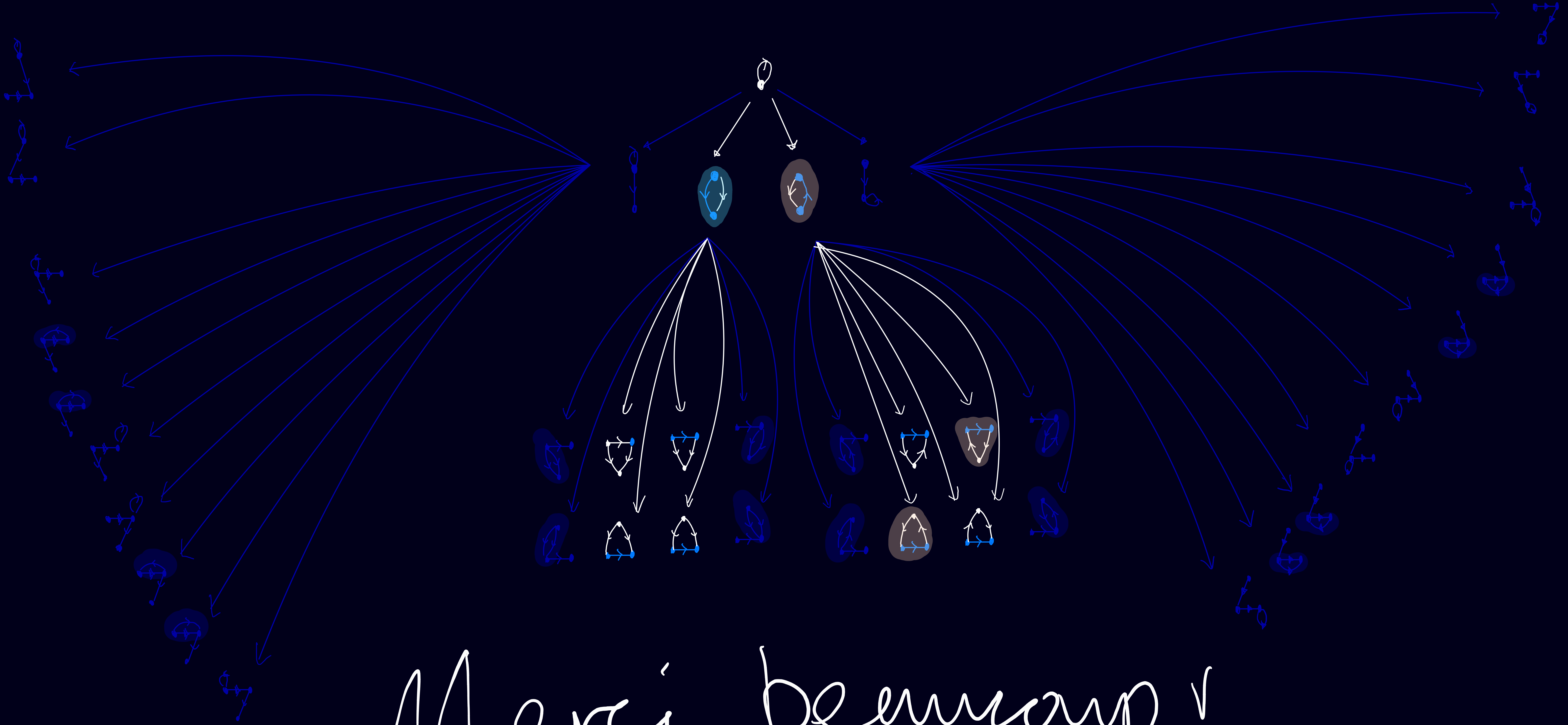
WHERE: $\forall$ $\begin{tikzcd} Y \arrow[d, "\cong"'] & & X \arrow[ll, bend right, "s"'] \arrow[dll, bend left, "s'"] \arrow[dl, phantom, "\Downarrow\gamma"] \\ Y' & & \end{tikzcd}$:

$$\begin{tikzcd} O \arrow[d] & I \arrow[l, "r"'] \arrow[d] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \sim_Y \begin{tikzcd} O \arrow[d] & I \arrow[l, "r"'] \arrow[d] \arrow[dl, phantom, "\Downarrow\gamma\cdot\alpha"] \\ Y' & X \arrow[l, "s'"] \end{tikzcd}$$

↳ DECATEGORIFICATION:

SUM OF COEFFICIENTS

$$\langle|\rho(\delta(r))|X\rangle := \left|\int^{Y\in\mathbb{D}_0}(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y\leftarrow\emptyset)\right|$$



Merci beaucoup!