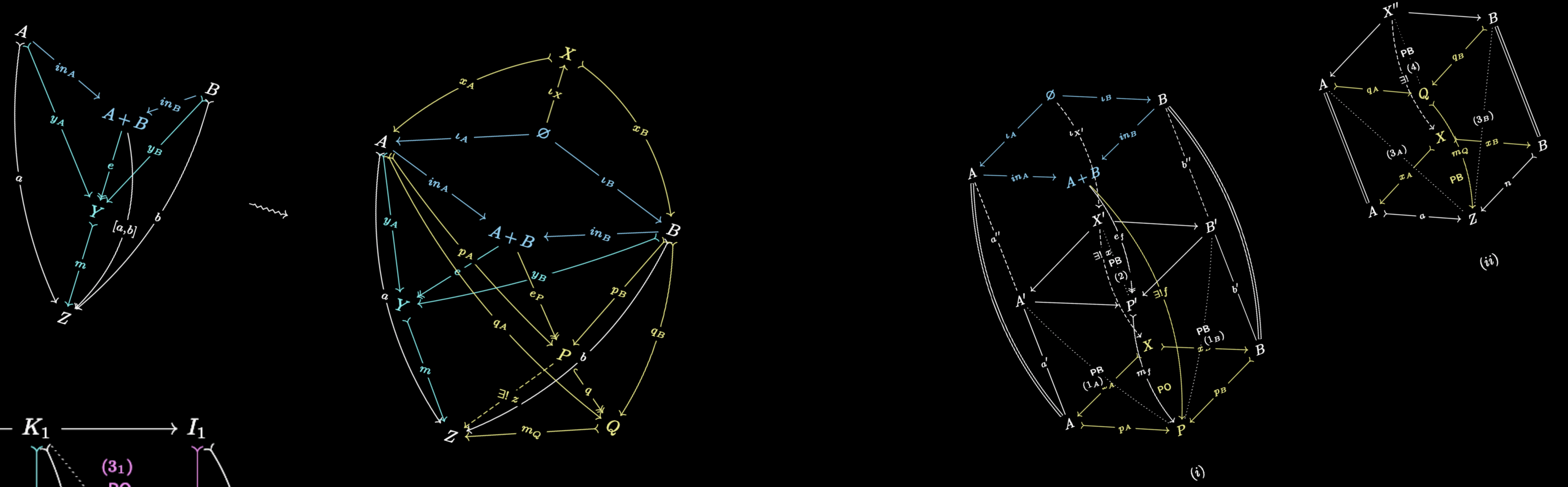
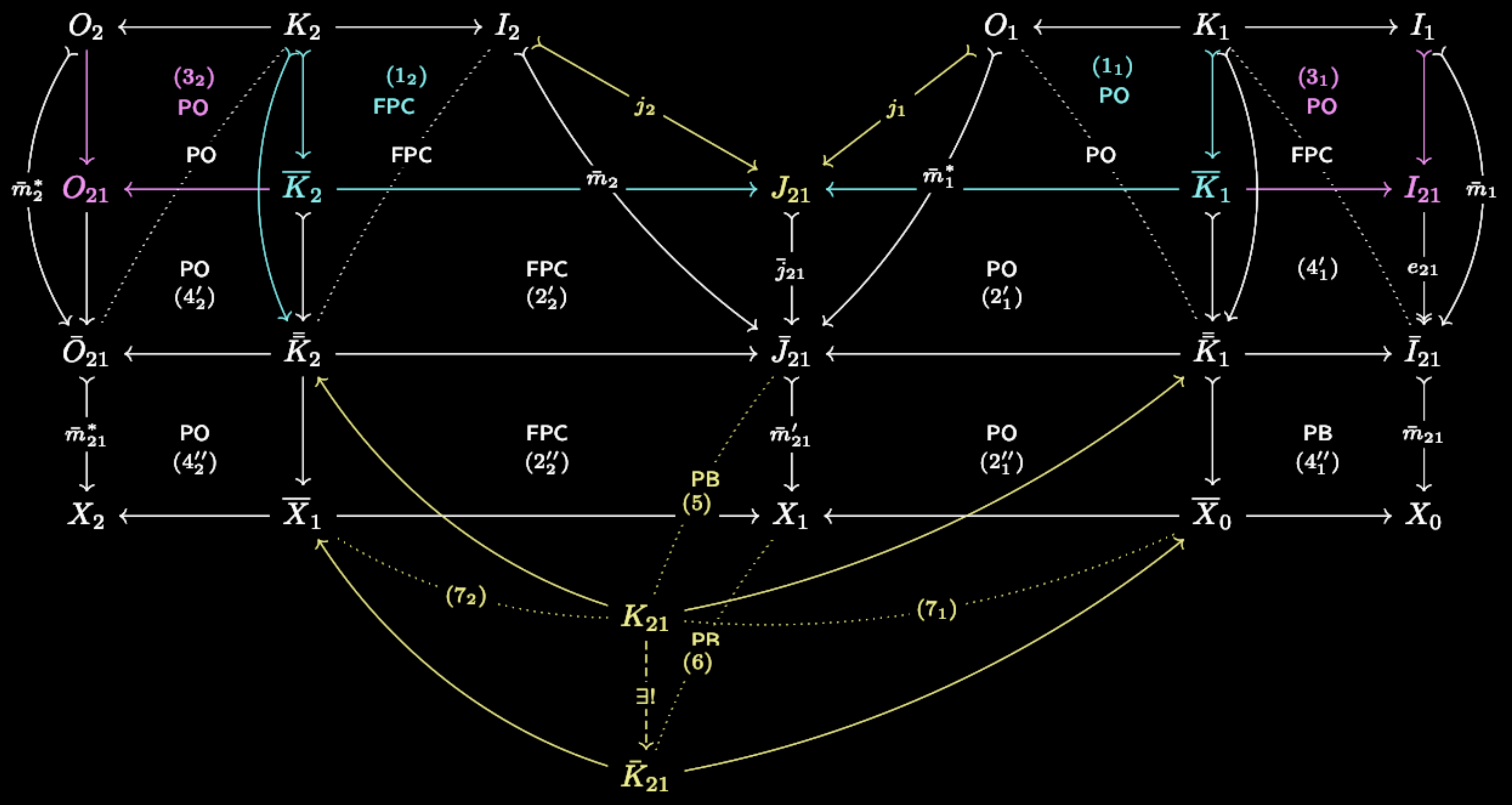
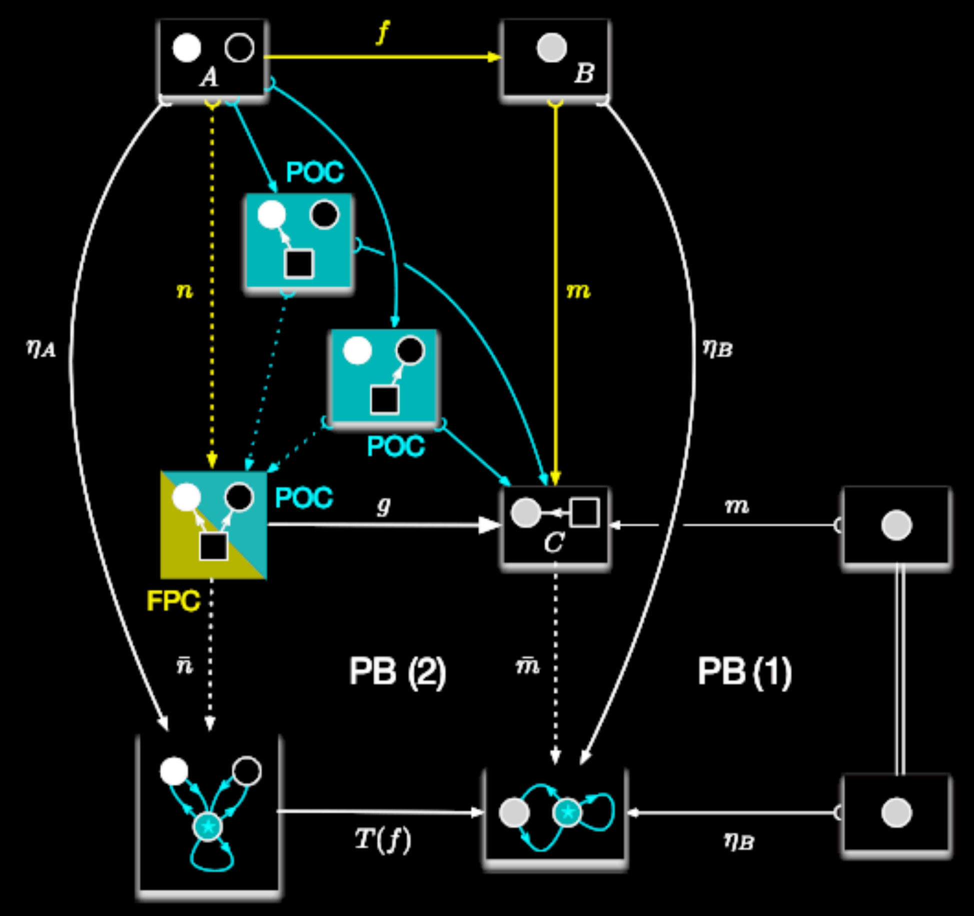
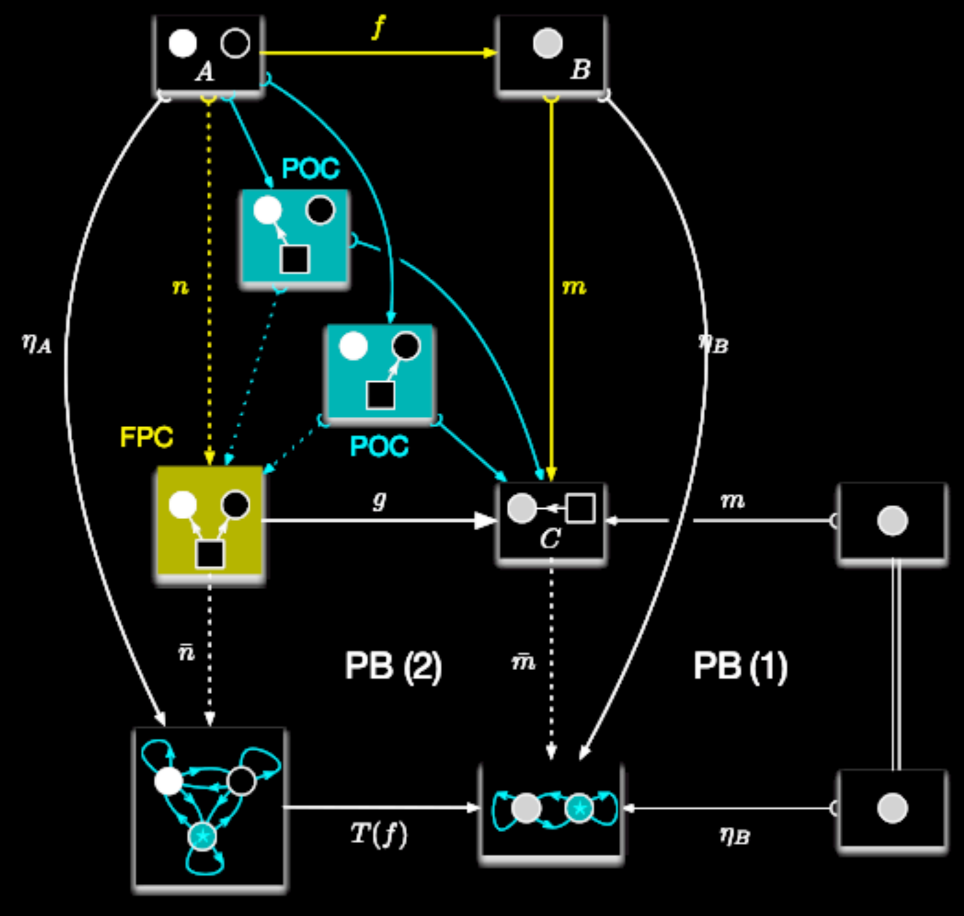
# icgt2021

14th International Conference on Graph Transformation
June 24–25 Bergen, Norway







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# Concurrency theorems for non-linear rewriting theories

*Joint work with* ***Jean Krivine (IRIF)*** *and* ***Russ Harmer (ENS Lyon)***

ICGT'21 (online), June 24, 2021


Nicolas Behr

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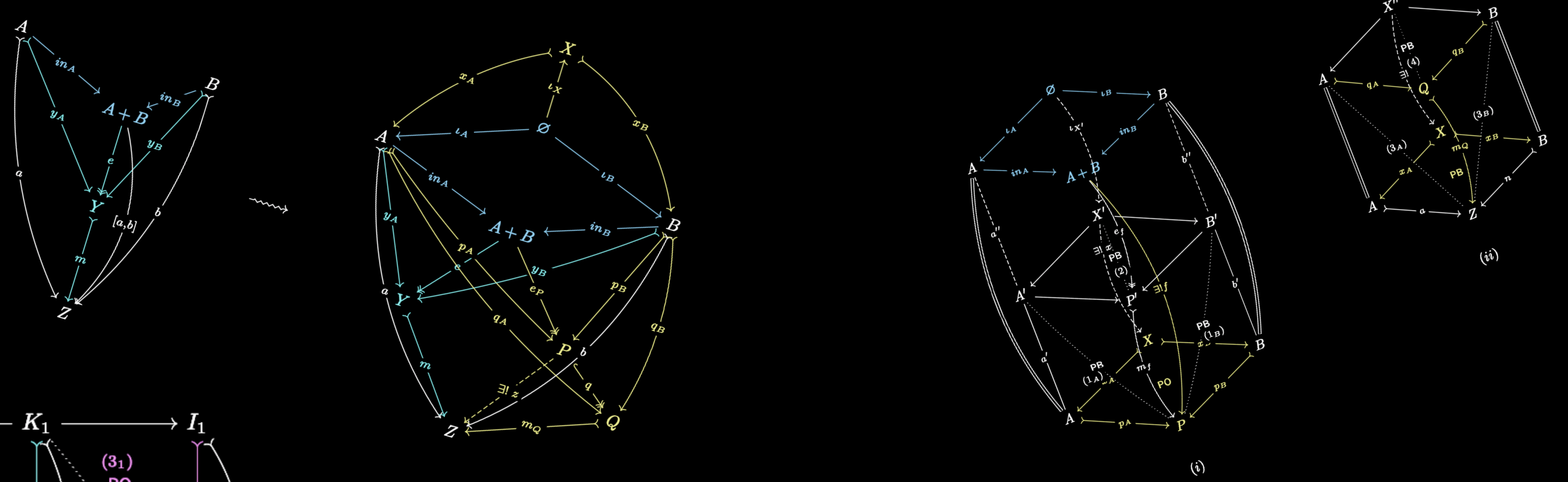

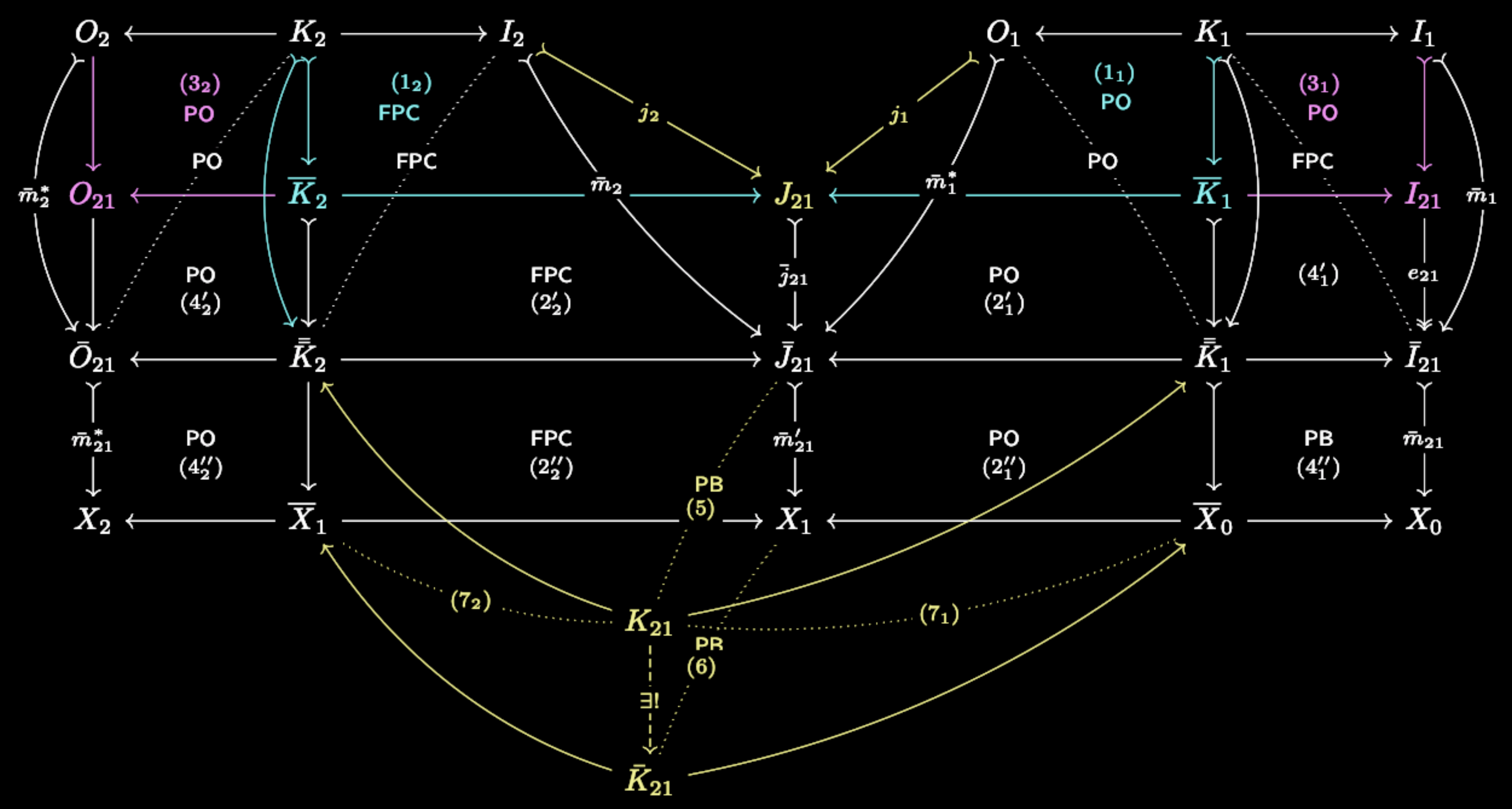

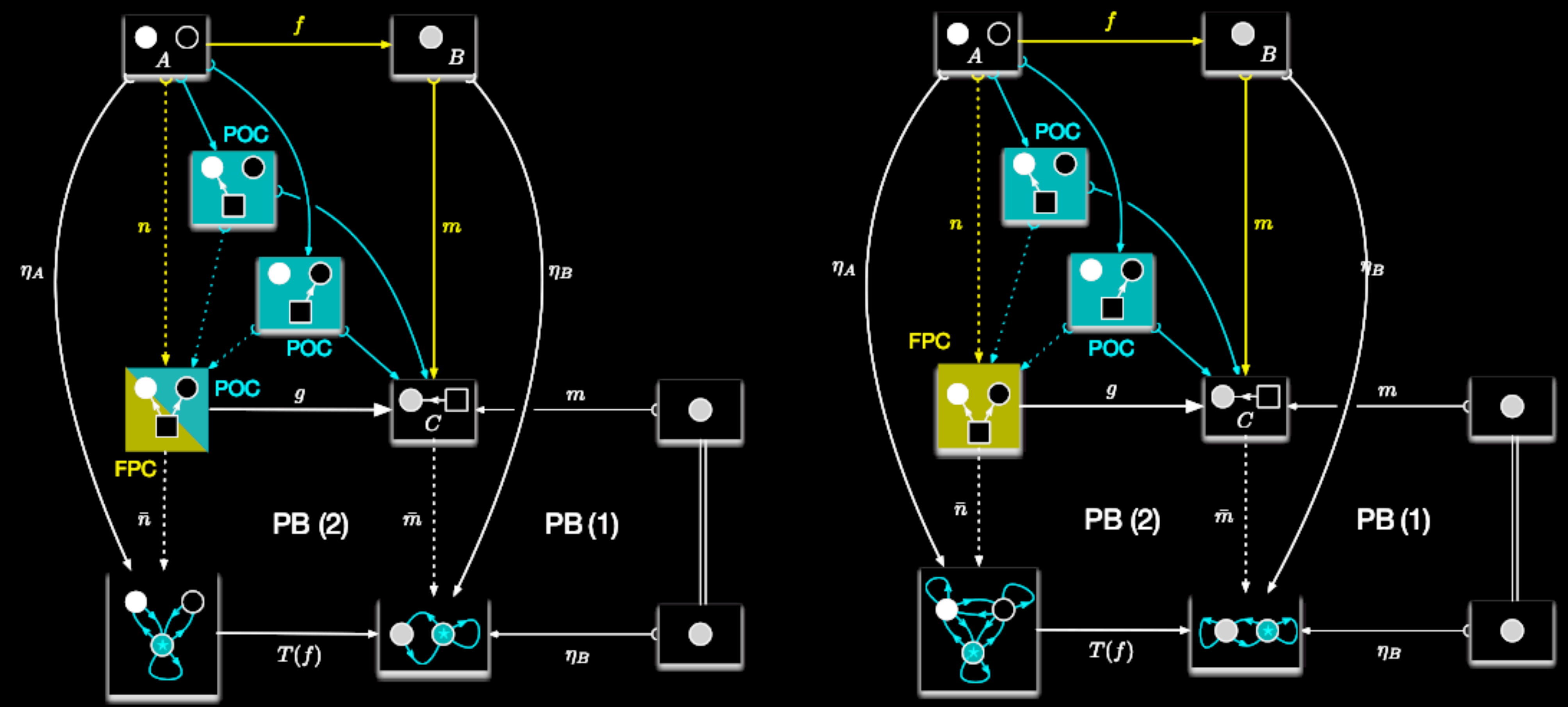

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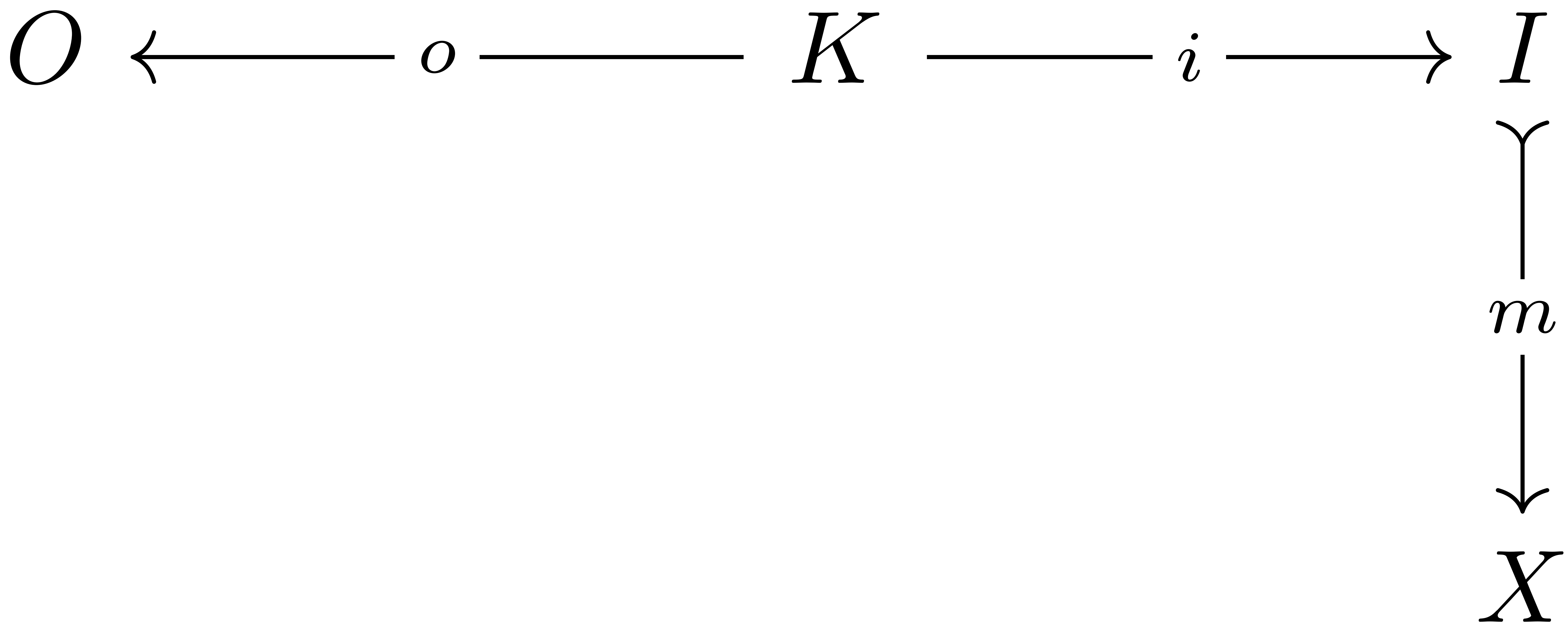
# Motivation 1: **non-linear Double Pushout (DPO) rewriting**



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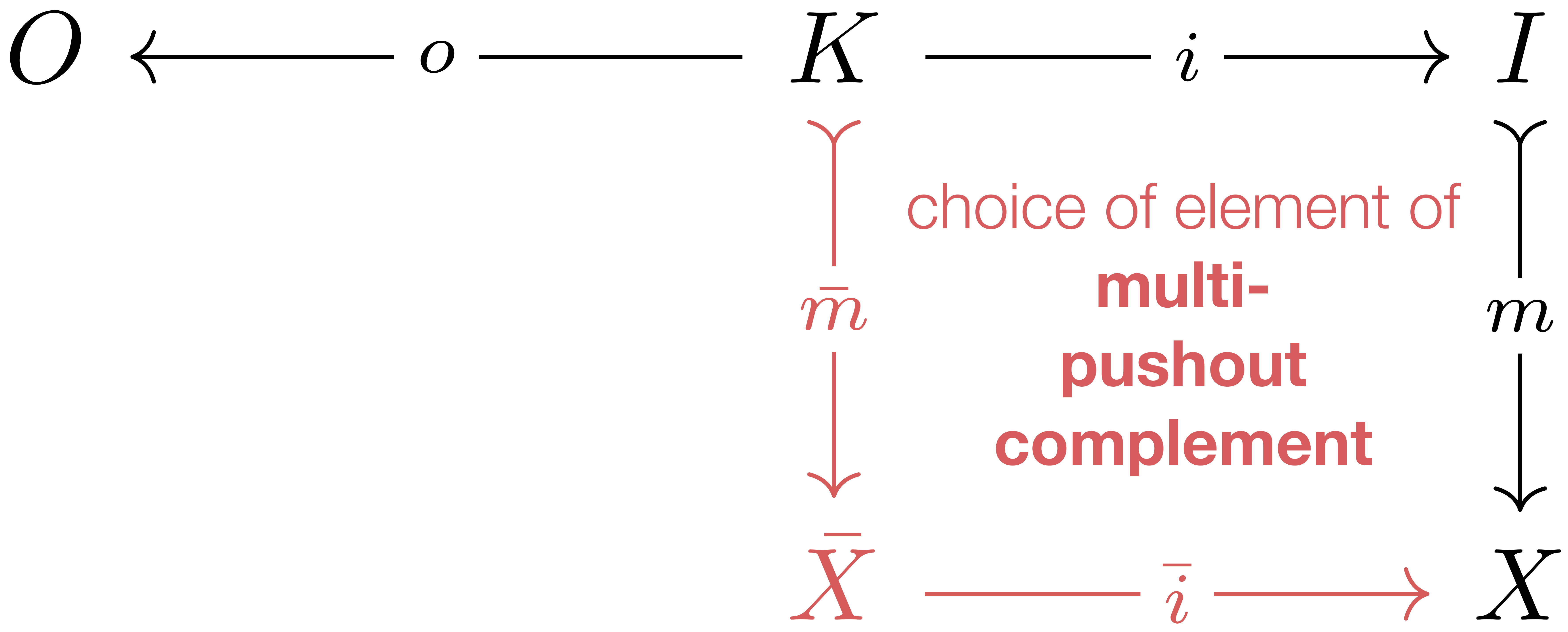

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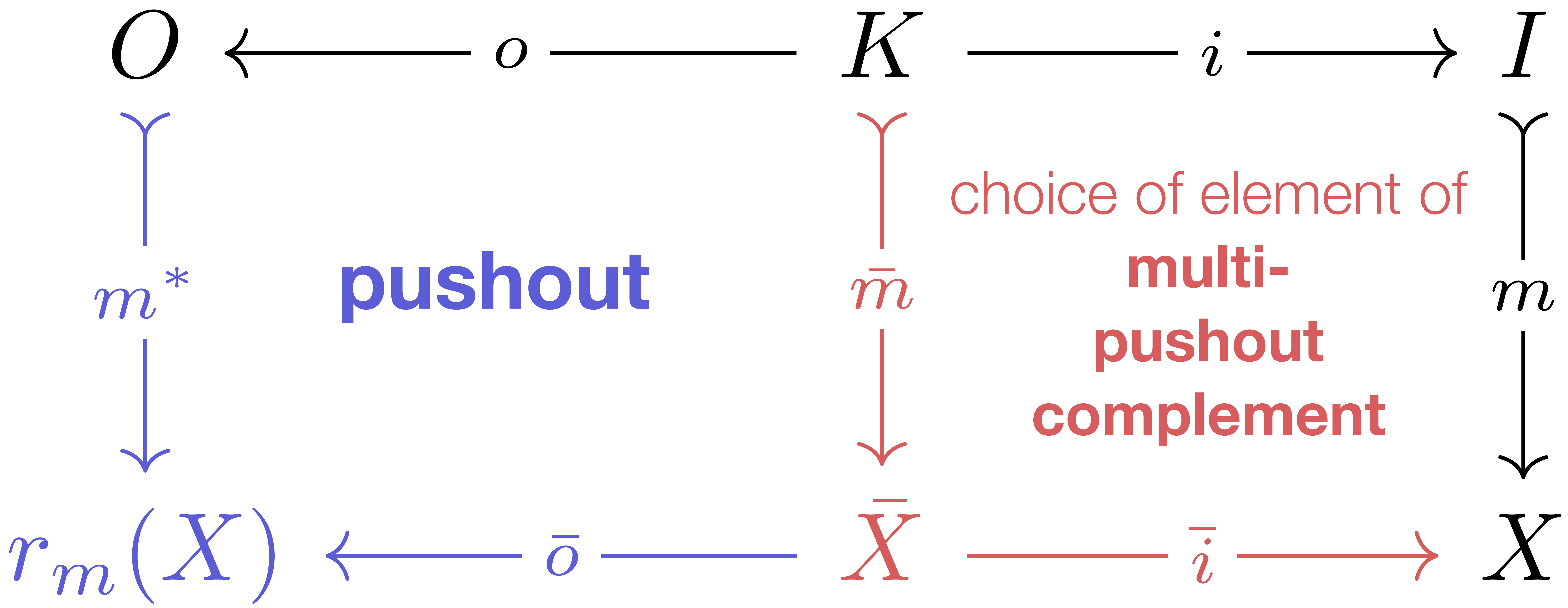

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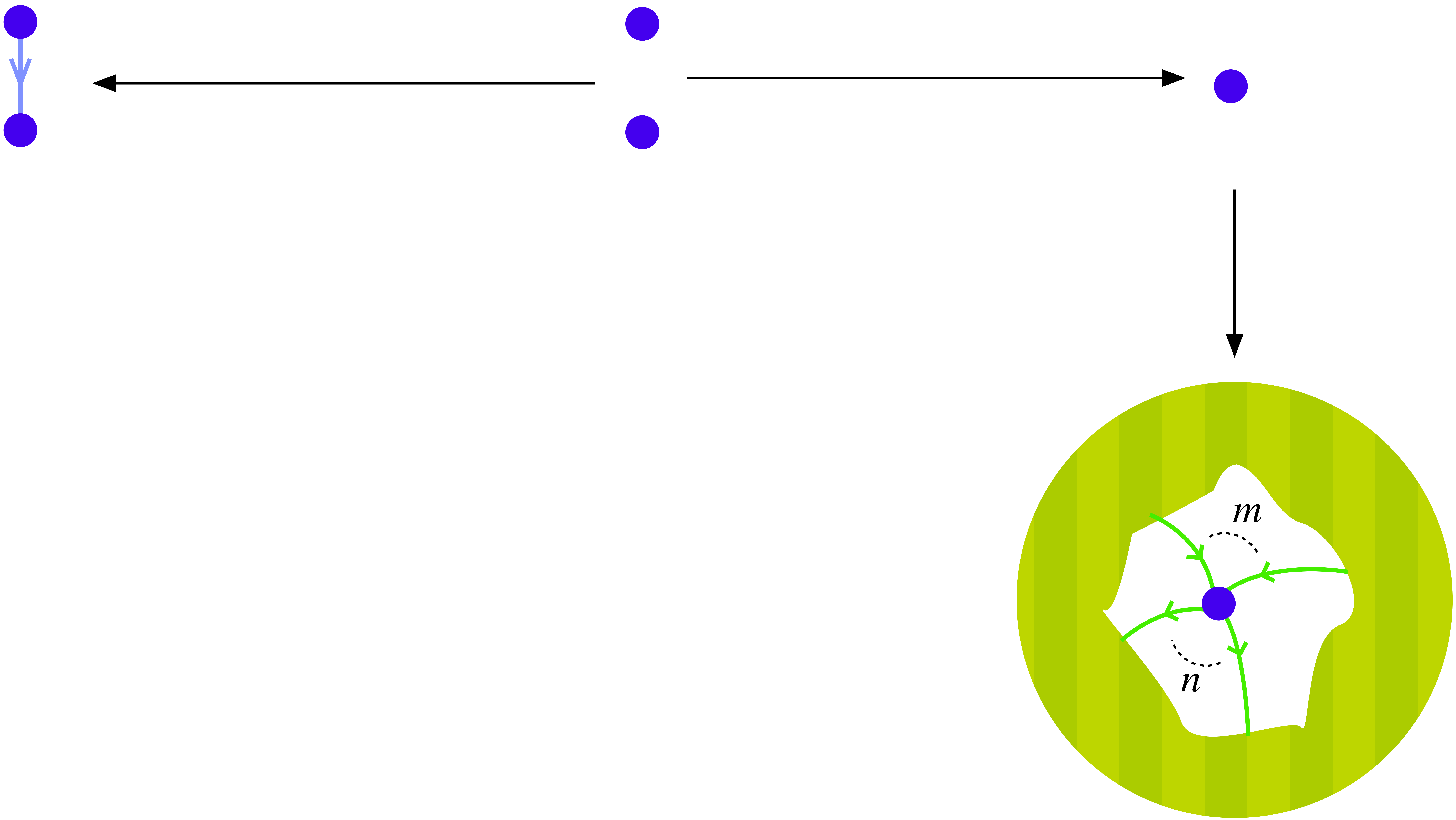

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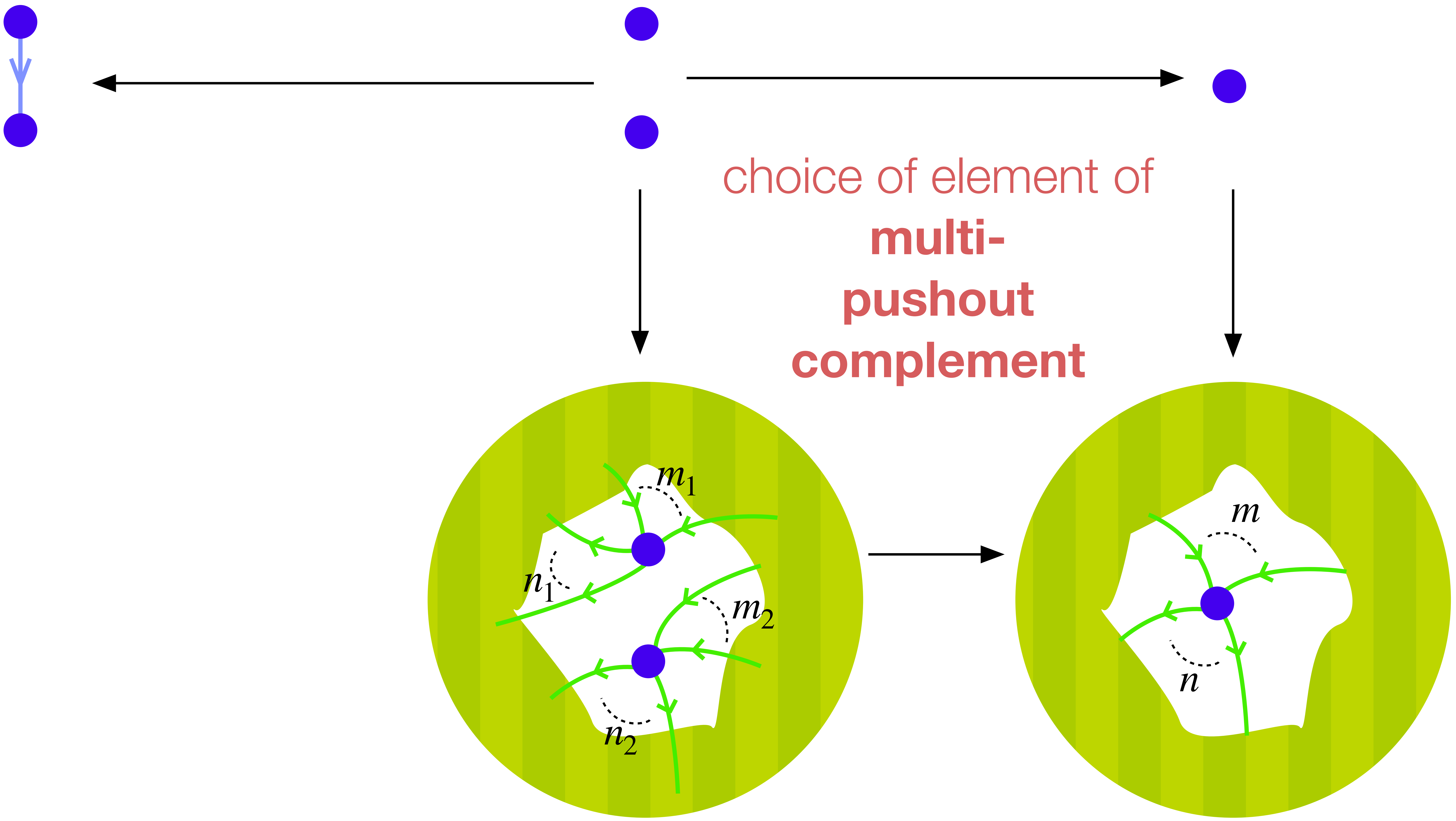

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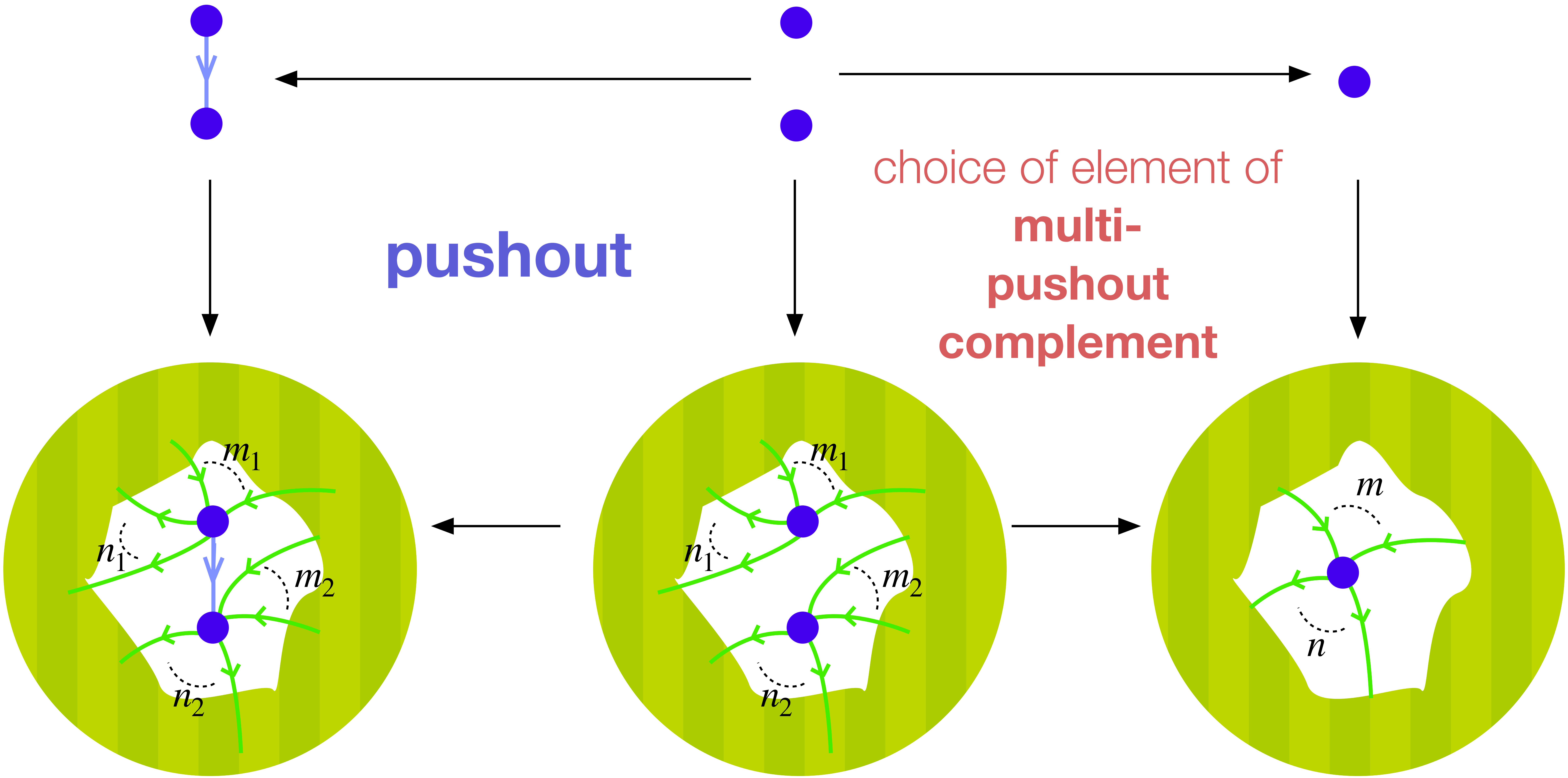

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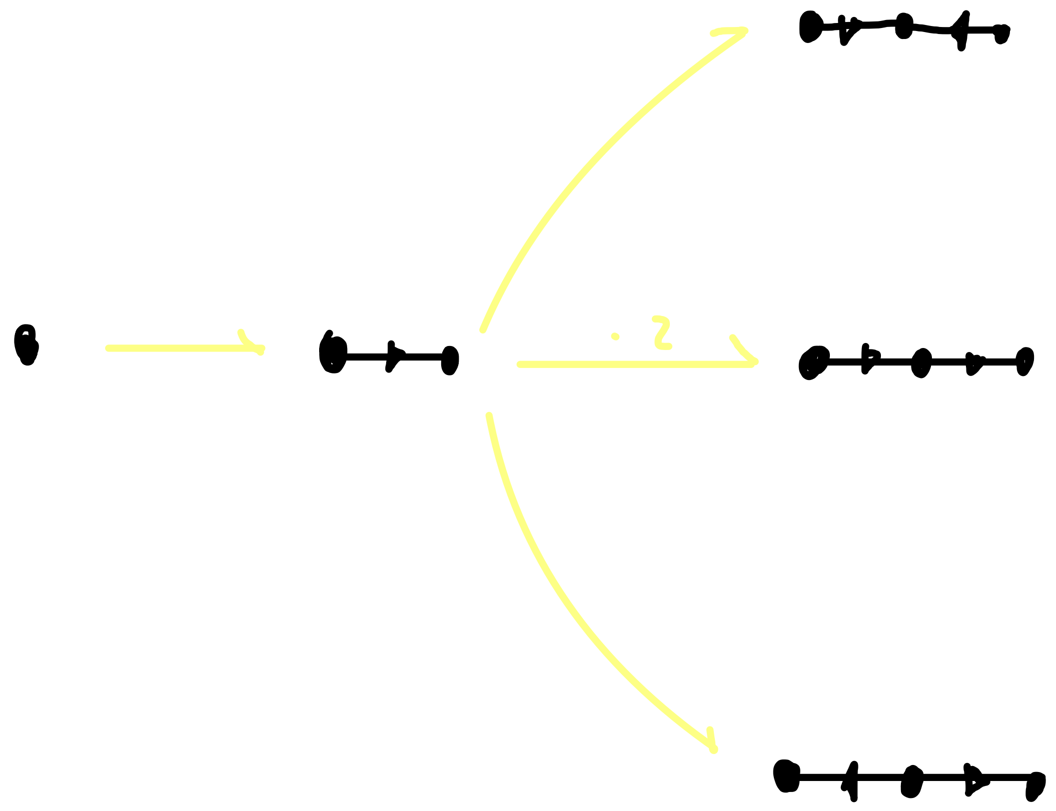

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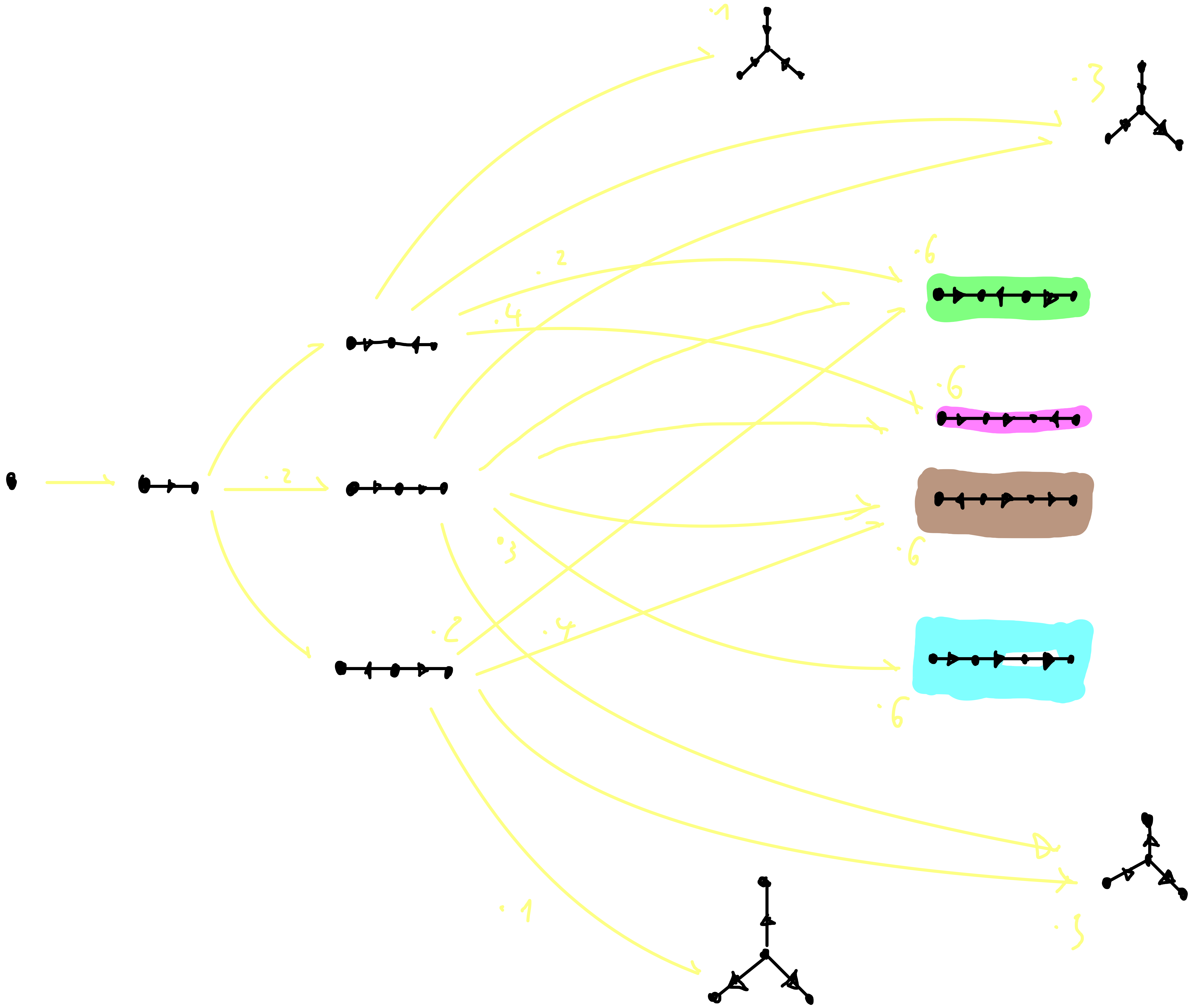
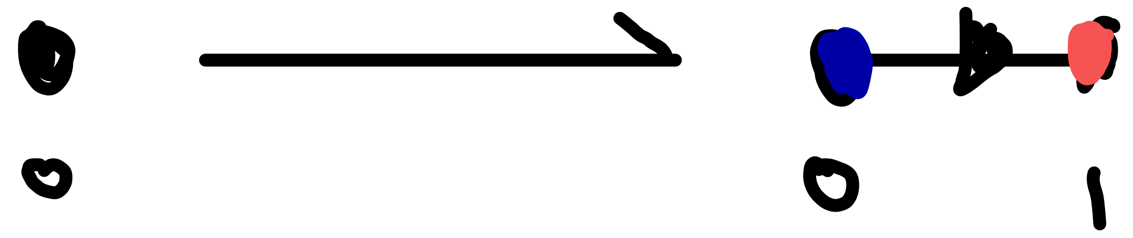


# Motivation 2: **non-linear Sesqui-Pushout (SqPO) rewriting**

Corradini, A., et al.: Sesqui-Pushout Rewriting. In: Graph Transformations. LNCS, vol. 4178, pp. 30–45. Springer Berlin Heidelberg (2006)

# Motivation 2: **non-linear Sesqui-Pushout (SqPO) rewriting**





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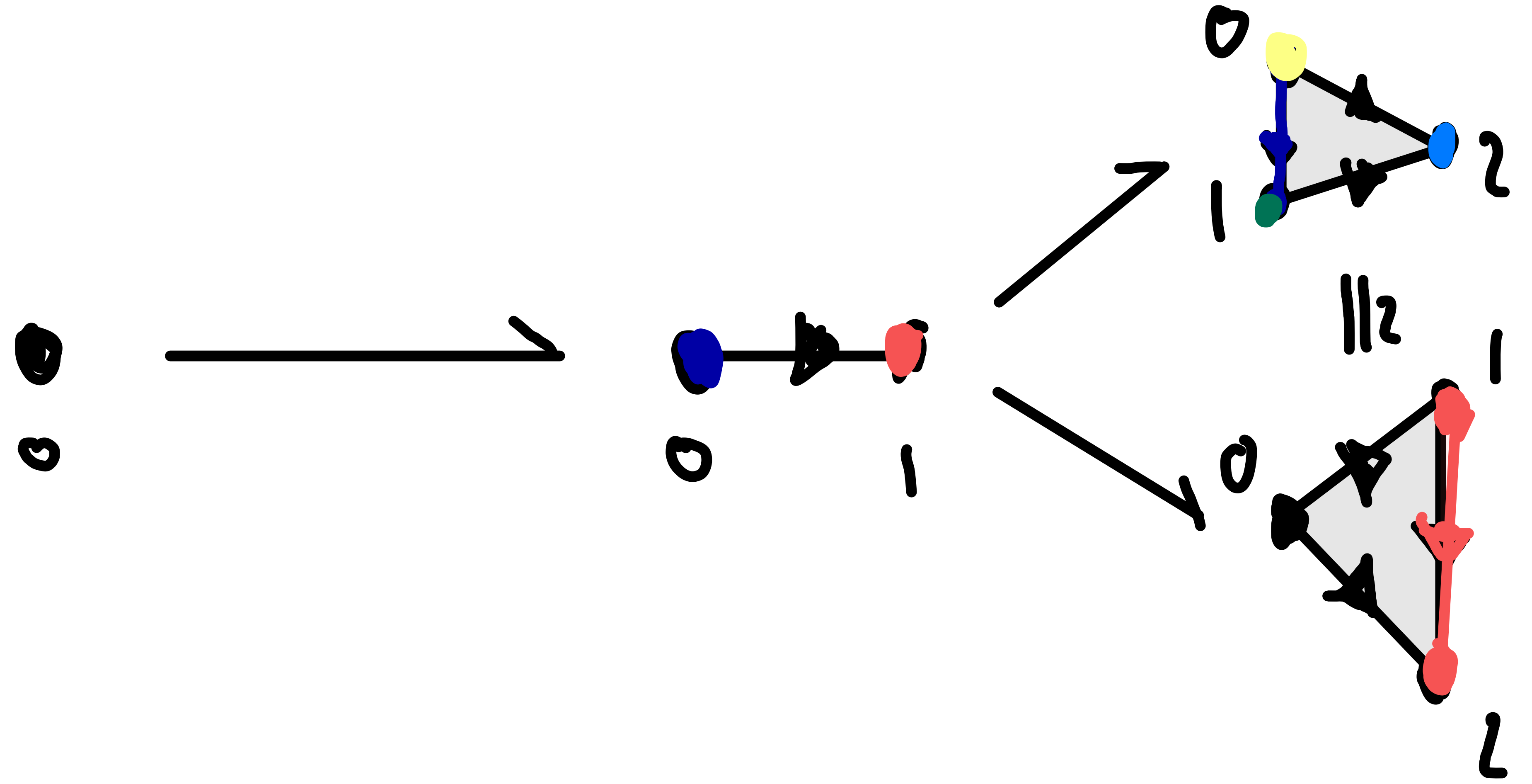


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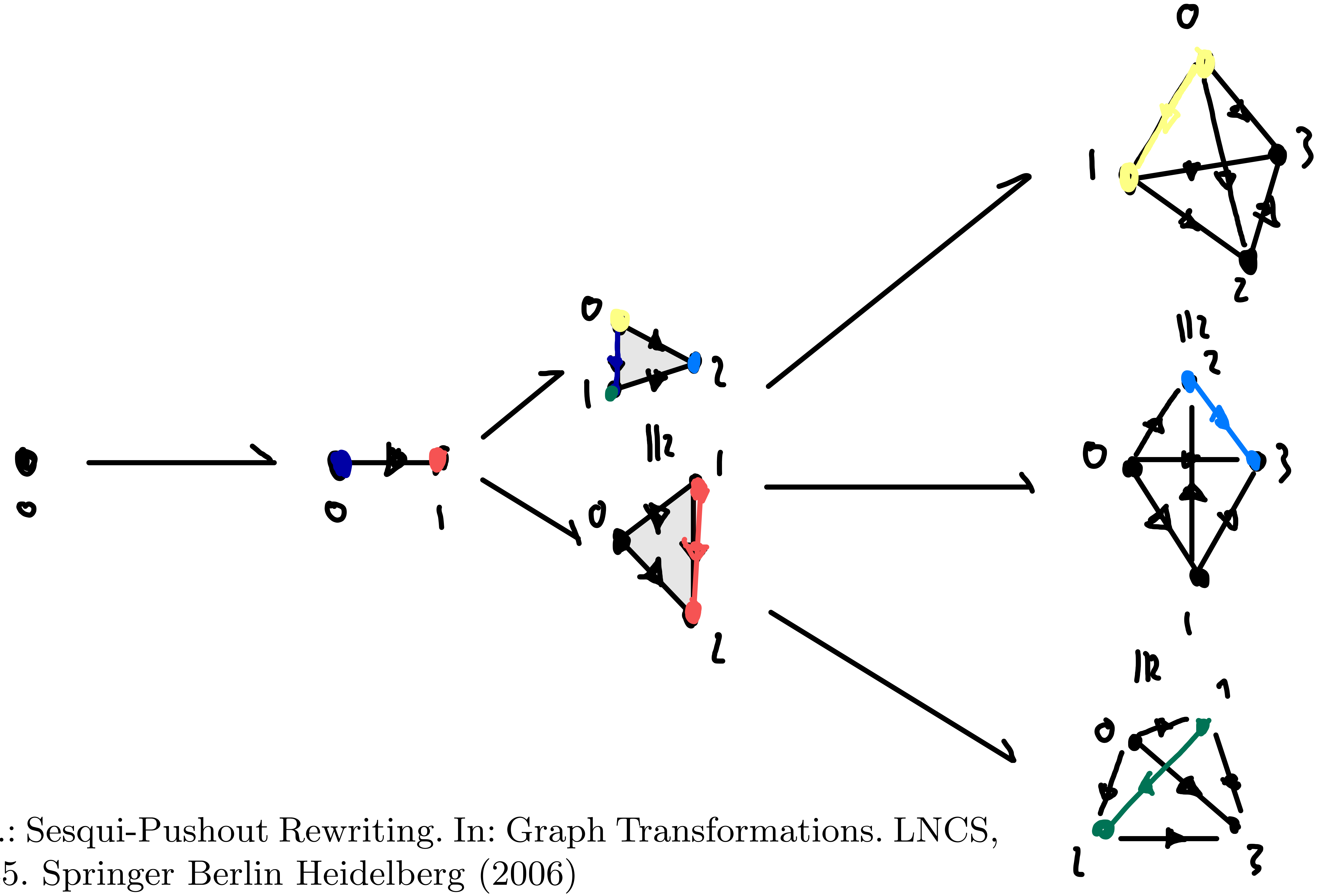

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# Motivation 3: **quasi-topoi and “natural” simple graph rewriting**

**simple graphs** as a *“bona fide”* rewriting semantics

$\Rightarrow$ requires the theory of **quasi-topoi** and of **non-linear SqPO-semantics** to be of any practical interest…

## Plan of the talk

1. **Quasi-topoi** in rewriting theory
2. **Prerequisites** for non-linear rewriting
3. **Non-linear DPO rewriting**
4. ***Non-linear SqPO rewriting***
5. Conclusion and outlook

# Plan of the talk

1. **Quasi-topoi** in rewriting theory
2. **Prerequisites** for non-linear rewriting
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## Definition

A category **C** is a quasi-topos iff

1. it has finite limits and colimits
2. it is locally Cartesian closed
3. it has a regular-subobject-classifier.

Johnstone, P.T., Lack, S., Sobociński, P.: Quasitoposes, Quasiadhesive Categories and Artin Glueing. In: Algebra and Coalgebra in Computer Science. LNCS, vol. 4624, pp. 312–326 (2007). https://doi.org/10.1007/978-3-540-73859-6_21

# **Quasi-topoi** — a natural setting for non-linear rewriting

## Proposition

Every quasi-topos $\mathbf{C}$ enjoys the following properties:

- It has (by definition) a stable system of monics $\mathcal{M} = \mathsf{rm}(\mathbf{C})$ (the class of regular monos), which coincides with the class of extremal monomorphisms, i.e., if $\mathsf{m} = \mathsf{f} \circ \mathsf{e}$ for $\mathsf{m} \in \mathsf{rm}(\mathbf{C})$ and $\mathsf{e} \in \mathsf{epi}(\mathbf{C})$, then $\mathsf{e} \in \mathsf{iso}(\mathbf{C})$.
- It has (by definition) a $\mathcal{M}$-partial map classifier $(\mathsf{T}, \eta)$.
- It is rm-quasi-adhesive, i.e., it has pushouts along regular monomorphisms, these are stable under pullbacks, and pushouts along regular monos are pullbacks.
- It is $\mathcal{M}$-adhesive.
- For all pairs of composable morphisms $\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{B}$ and $\mathsf{B} \xrightarrow{\mathsf{m}} \mathsf{C}$ with $\mathsf{m} \in \mathcal{M}$, there exists a final pullback-complement (FPC) $\mathsf{A} \xrightarrow{\mathsf{n}} \mathsf{F} \xrightarrow{\mathsf{g}} \mathsf{C}$, and with $\mathsf{n} \in \mathcal{M}$.
- It possesses an epi-$\mathcal{M}$-factorization: each morphism $\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{B}$ factors as $\mathsf{f} = \mathsf{m} \circ \mathsf{e}$, with morphisms $\mathsf{A} \xrightarrow{\mathsf{e}} \mathsf{B}$ in $\mathsf{epi}(\mathbf{C})$ and $\mathsf{B} \xrightarrow{\mathsf{m}} \mathsf{A}$ in $\mathcal{M}$ (uniquely up to isomorphism in B).

# **Quasi-topoi** — a natural setting for non-linear rewriting

## Definition

The category Graph of directed multigraphs is defined as the presheaf category **Graph** $:= (\mathbb{G}^{op} \to \mathbf{Set})$, where $\mathbb{G} := (\cdot \rightrightarrows \star)$ is a category with two objects and two morphisms.

- Objects $G = (V_G, E_G, s_G, t_G)$ of **Graph** are given by a set of vertices $V_G$, a set of directed edges $E_G$ and the source and target functions $s_G, t_G : E_G \to V_G$.
- Morphisms between $G, H \in obj(\mathbf{Graph})$ are of the form $\varphi = (\varphi_V, \varphi_E)$, with $\varphi_V : V_G \to V_H$ and $\varphi_E : E_G \to E_H$ such that $\varphi_V \circ s_G = s_H \circ \varphi_E$ and $\varphi_V \circ t_G = t_H \circ \varphi_E$.

# **Quasi-topoi** — a natural setting for non-linear rewriting

## Definition

The category $\mathbf{SGraph}$ of directed simple graphs is defined as the category of binary relations $\mathbf{BRel} \cong \mathbf{Set} /\!\!/ \Delta$. Here, $\Delta : \mathbf{Set} \to \mathbf{Set}$ is the pullback-preserving diagonal functor defined via $\Delta X := X \times X$, and $\mathbf{Set} /\!\!/ \Delta$ denotes the full subcategory of the slice category $\mathbf{Set}/\Delta$ defined via restriction to objects $m : X \to \Delta X$ that are monomorphisms.

- An object of $\mathbf{SGraph}$ is given by $S = (V, E, \iota)$, where $V$ is a set of vertices, $E$ is a set of directed edges, and where $\iota : E \to V \times V$ is an injective function.
- A morphism $f = (f_V, f_E)$ between objects $S$ and $S'$ is a pair of functions $f_V : V \to V'$ and $f_E : E \to E'$ such that $\iota' \circ f_E = (f_V \times f_V) \circ \iota$.

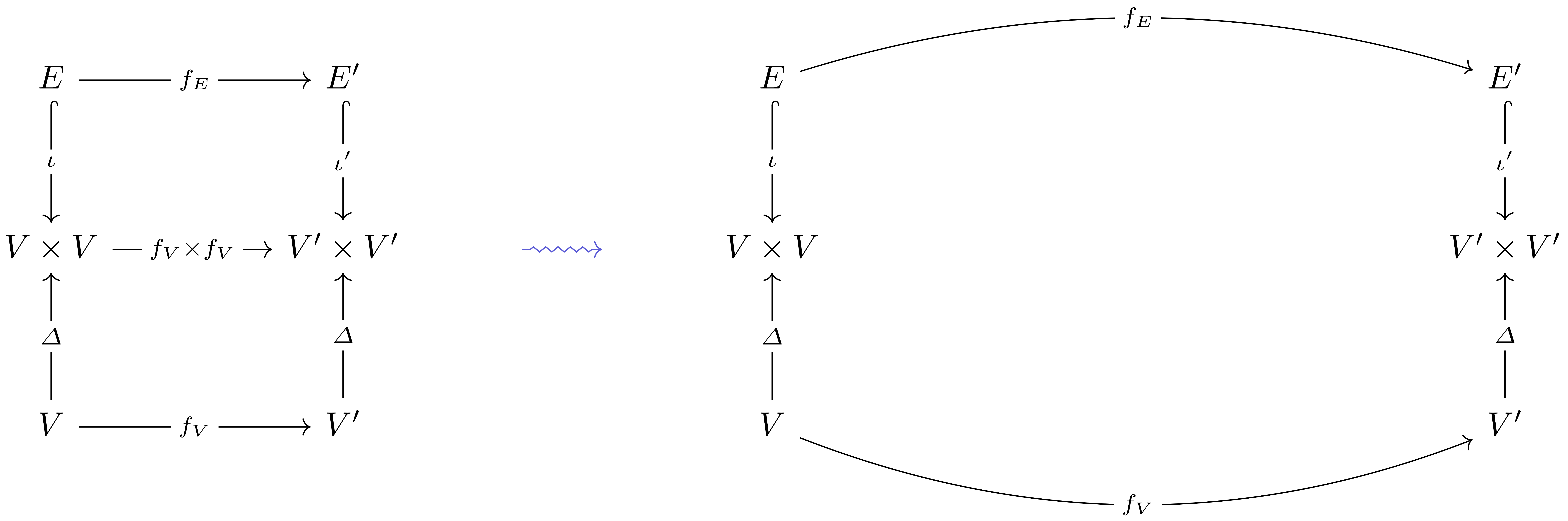
# **Quasi-topoi** — a natural setting for non-linear rewriting

$$
\begin{array}{c}
E \\
\Big\downarrow{\scriptstyle \iota} \\
V \times V \\
\Big\uparrow{\scriptstyle \Delta} \\
V
\end{array}
$$

# **Quasi-topoi** — a natural setting for non-linear rewriting

$$
\begin{array}{ccc}
E & \xrightarrow{\quad f_E \quad} & E' \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\
V \times V & \xrightarrow{\; f_V \times f_V \;} & V' \times V' \\
{\scriptstyle \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\
V & \xrightarrow{\quad f_V \quad} & V'
\end{array}
$$

# **Quasi-topoi** — a natural setting for non-linear rewriting



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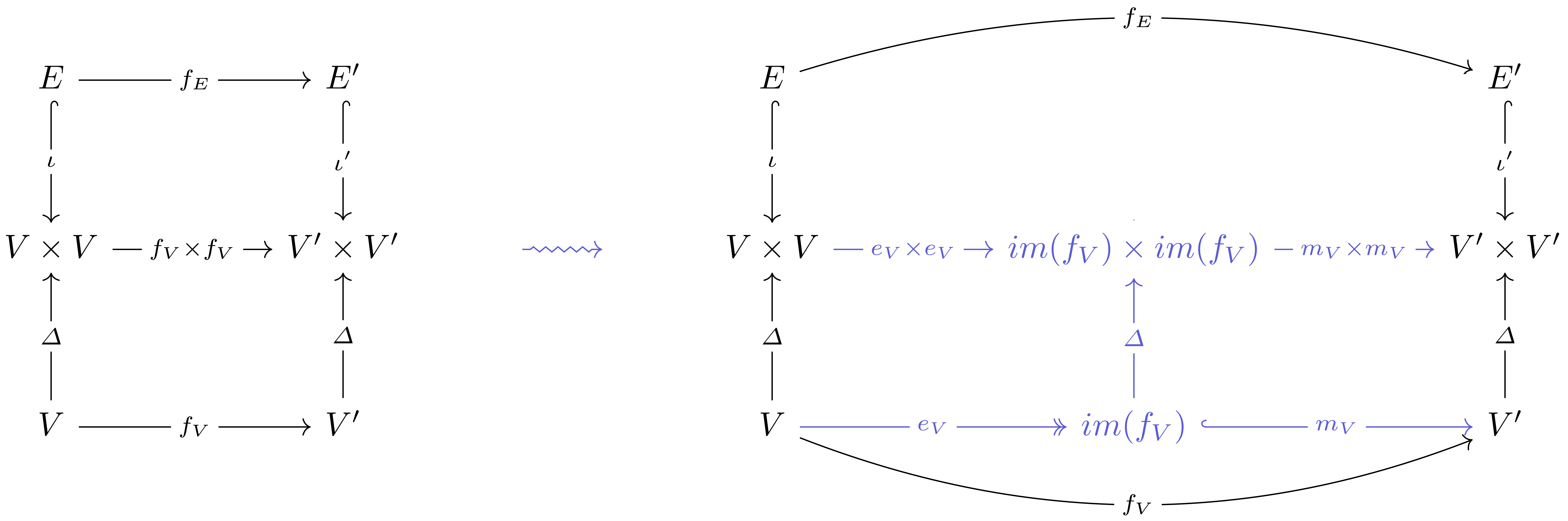

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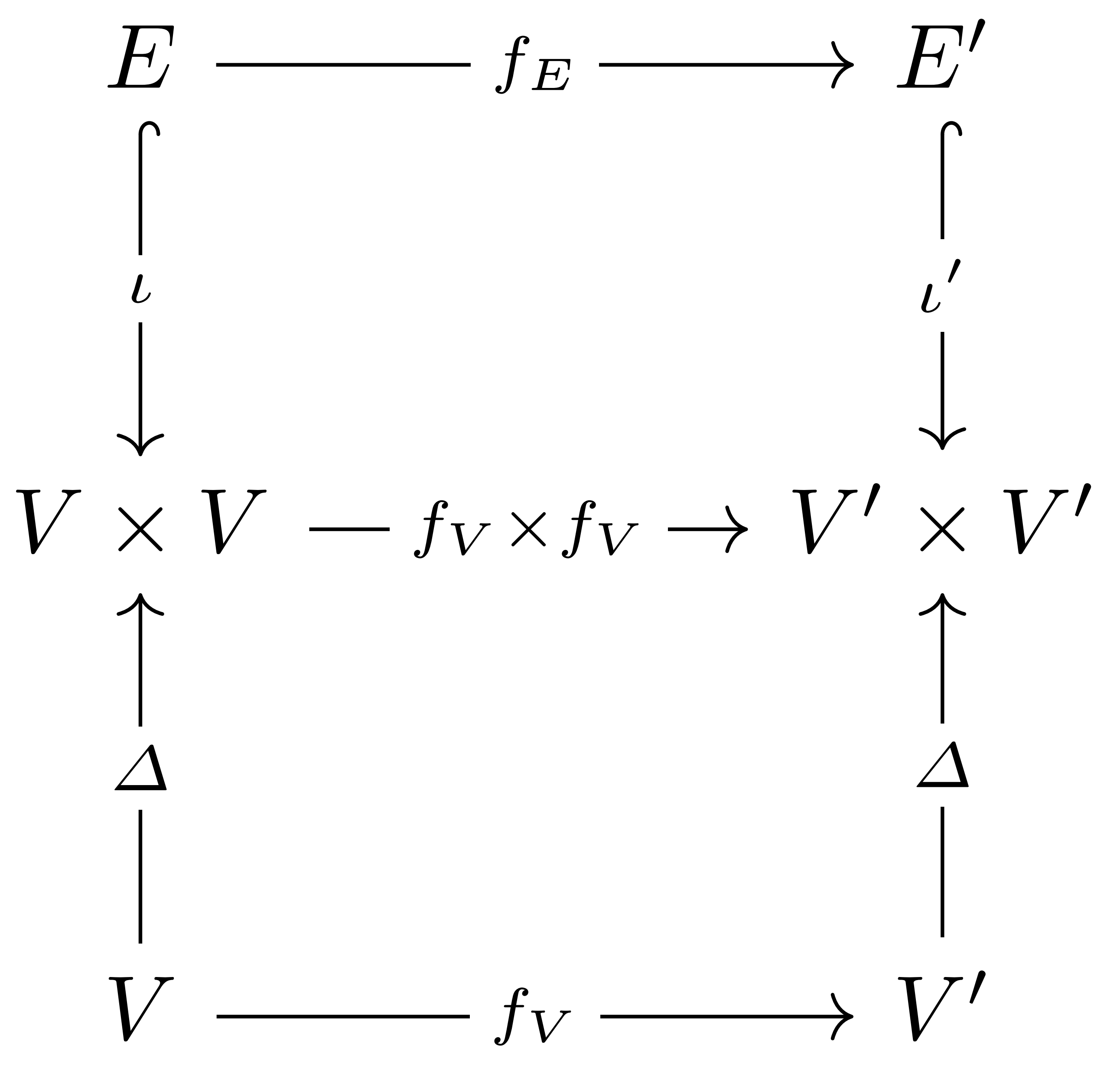


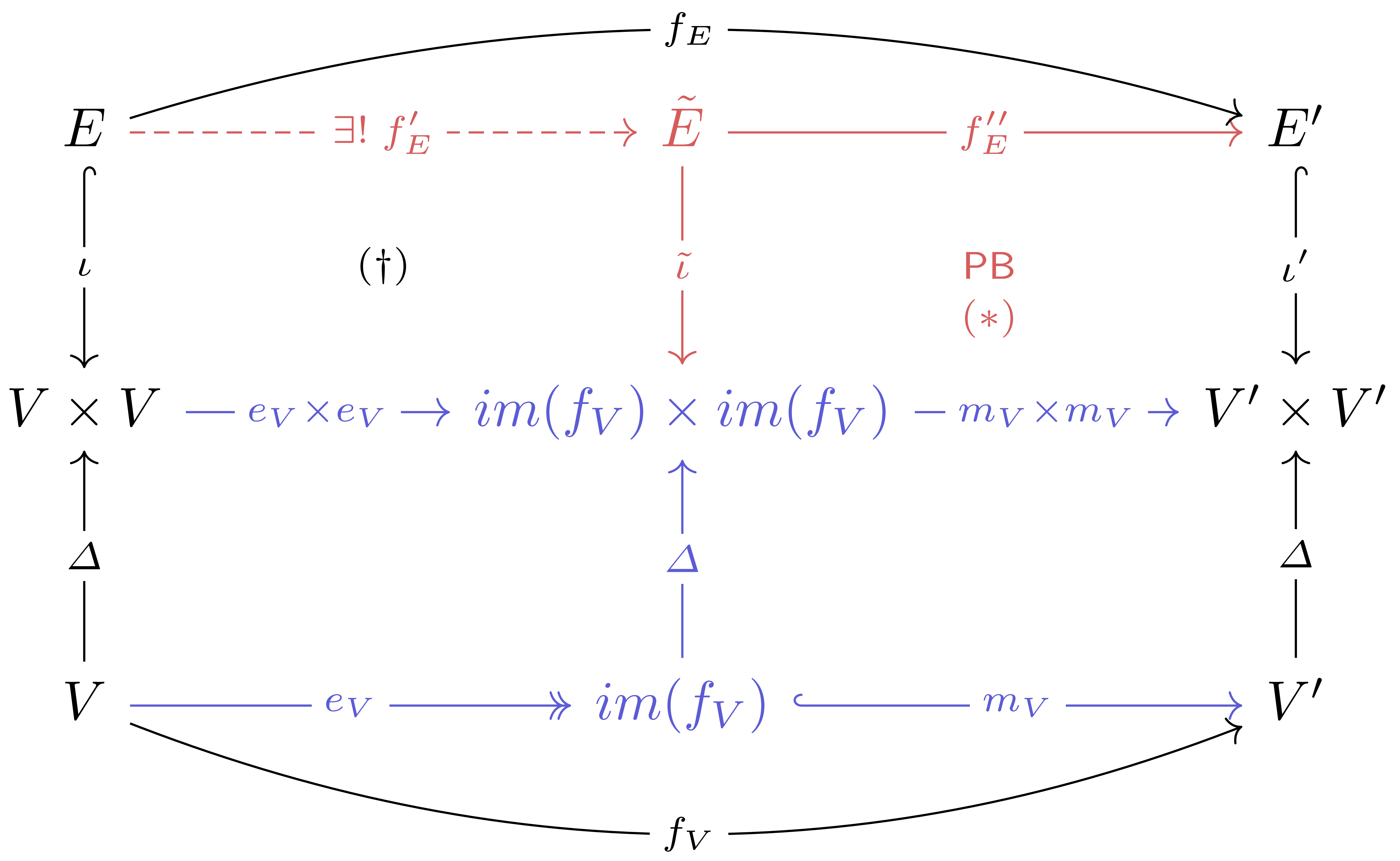

# (M-) partial map classifiers

## Definition

For a category $\mathbf{C}$, a stable system of monics $\mathcal{M}$ is a class of monomorphisms of $\mathbf{C}$ that (i) includes all isomorphisms, (ii) is stable under composition, and (iii) is stable under pullbacks (i.e., if $(f', m')$ is a pullback of $(m, f)$ with $m \in \mathcal{M}$, then $m' \in \mathcal{M}$).We will reserve the notation $\rightarrowtail$ for monics in $\mathcal{M}$, and $\hookrightarrow$ for generic monics.

Corradini, A., et al.: AGREE – Algebraic Graph Rewriting with Controlled Embedding. In: Graph Transformation (ICGT 2015). LNCS, vol. 9151, pp. 35–51. Cham (2015). https://doi.org/10.1007/978-3-319-21145-9_3

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## Definition

For a stable system of monics $\mathcal{M}$ in a category $\mathsf{C}$, an $\mathcal{M}$-partial map classifier $(\mathsf{T}, \eta)$ is a functor $\mathsf{T} : \mathsf{C} \to \mathsf{C}$ and a natural transformation $\eta : \mathsf{ID}_{\mathsf{C}} \dot{\to} \mathsf{T}$ such that

1. for all $\mathsf{X} \in \mathtt{obj}(\mathsf{C})$, $\eta_{\mathsf{X}} : \mathsf{X} \to \mathsf{T}(\mathsf{X})$ is in $\mathcal{M}$
2. for each span $(\mathsf{A} \xleftarrow{\mathsf{m}} \mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{B})$ with $\mathsf{m} \in \mathcal{M}$, there exists a unique morphism $\mathsf{A} \xrightarrow{\varphi(\mathsf{m},\mathsf{f})} \mathsf{T}(\mathsf{B})$ such that $(\mathsf{m}, \mathsf{f})$ is a pullback of $(\varphi(\mathsf{m}, \mathsf{f}), \eta_{\mathsf{B}})$.

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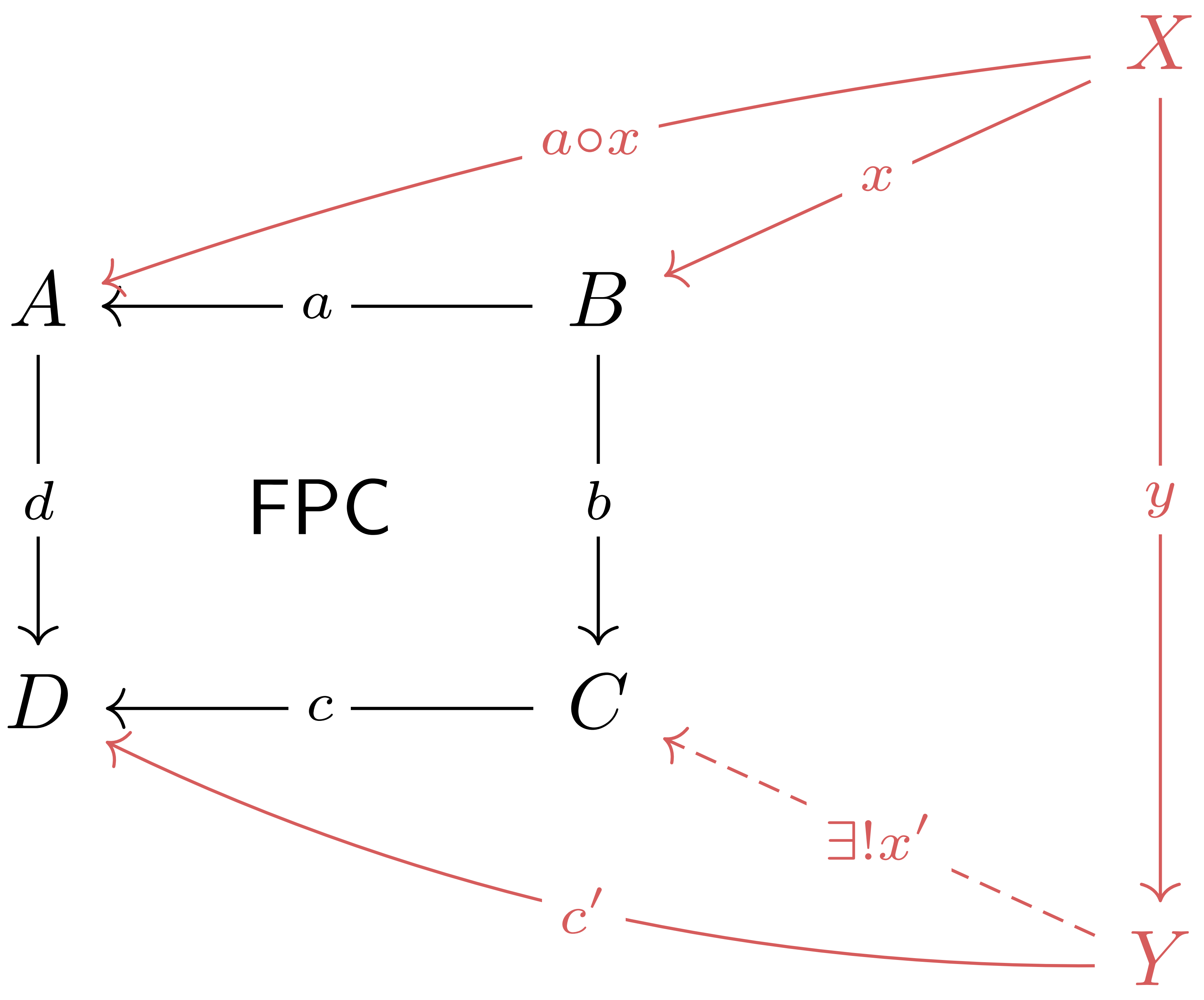
# Final pullback complements (FPCs) via partial map classifiers

## Universal property of final pullback complements (FPCs)

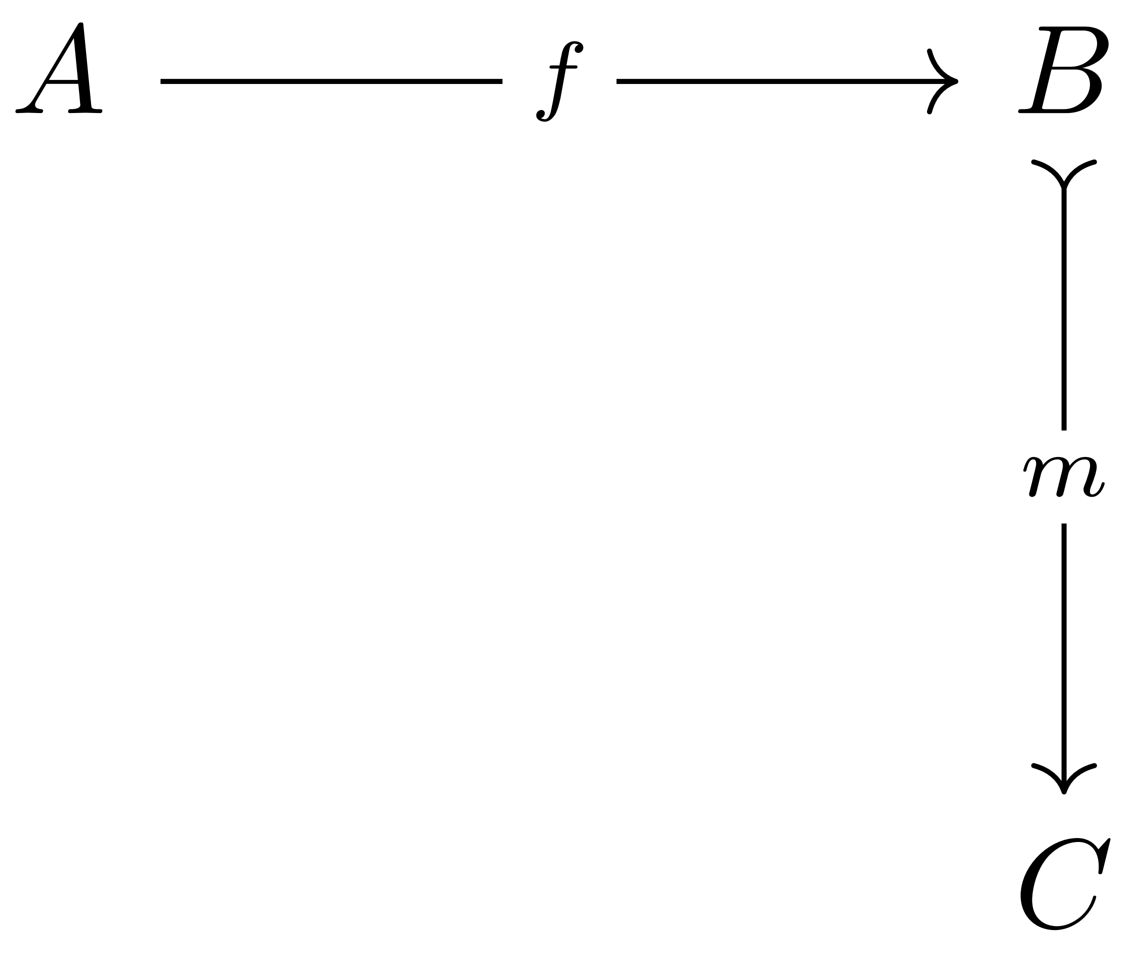
Given a commutative diagram as below, where $(a \circ x.y)$ is a pullback of $(d, c')$, there exists a morphism $Y - x' \rightarrow C$ that is unique up to isomorphisms, and which satisfies that $(x, y)$ is the PB of $(b, x')$.



# Final pullback complements (FPCs) via partial map classifiers

## Theorem

For a category **C** with $\mathcal{M}$-partial map classifier $(\mathsf{T}, \eta)$, the final pullback complement (FPC) of a composable sequence of arrows $\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{B}$ and $\mathsf{B} \xrightarrow{\mathsf{m}} \mathsf{C}$ with $\mathsf{m} \in \mathcal{M}$ is guaranteed to exist, and is constructed via the following algorithm:



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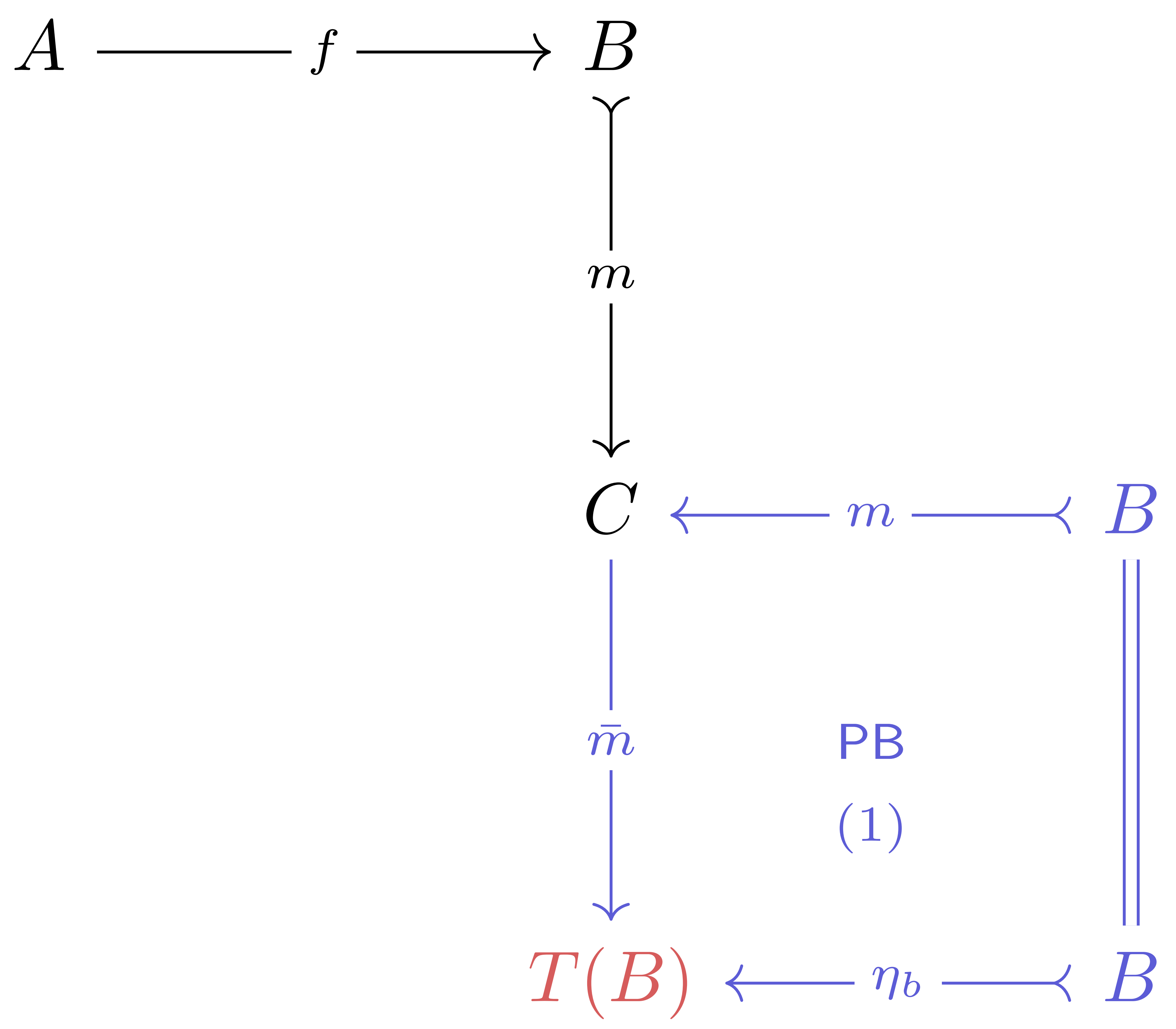
1. Let $\bar{\mathsf{m}} := \varphi(\mathsf{m}, \mathsf{id_B})$ (i.e., the morphism that exists by the universal property of $(\mathsf{T}, \eta)$, cf. square (1) below).



# Final pullback complements (FPCs) via partial map classifiers

## Theorem

For a category $\mathbf{C}$ with $\mathcal{M}$-partial map classifier $(T, \eta)$, the final pullback complement (FPC) of a composable sequence of arrows $A \xrightarrow{f} B$ and $B \xrightarrow{m} C$ with $m \in \mathcal{M}$ is guaranteed to exist, and is constructed via the following algorithm:

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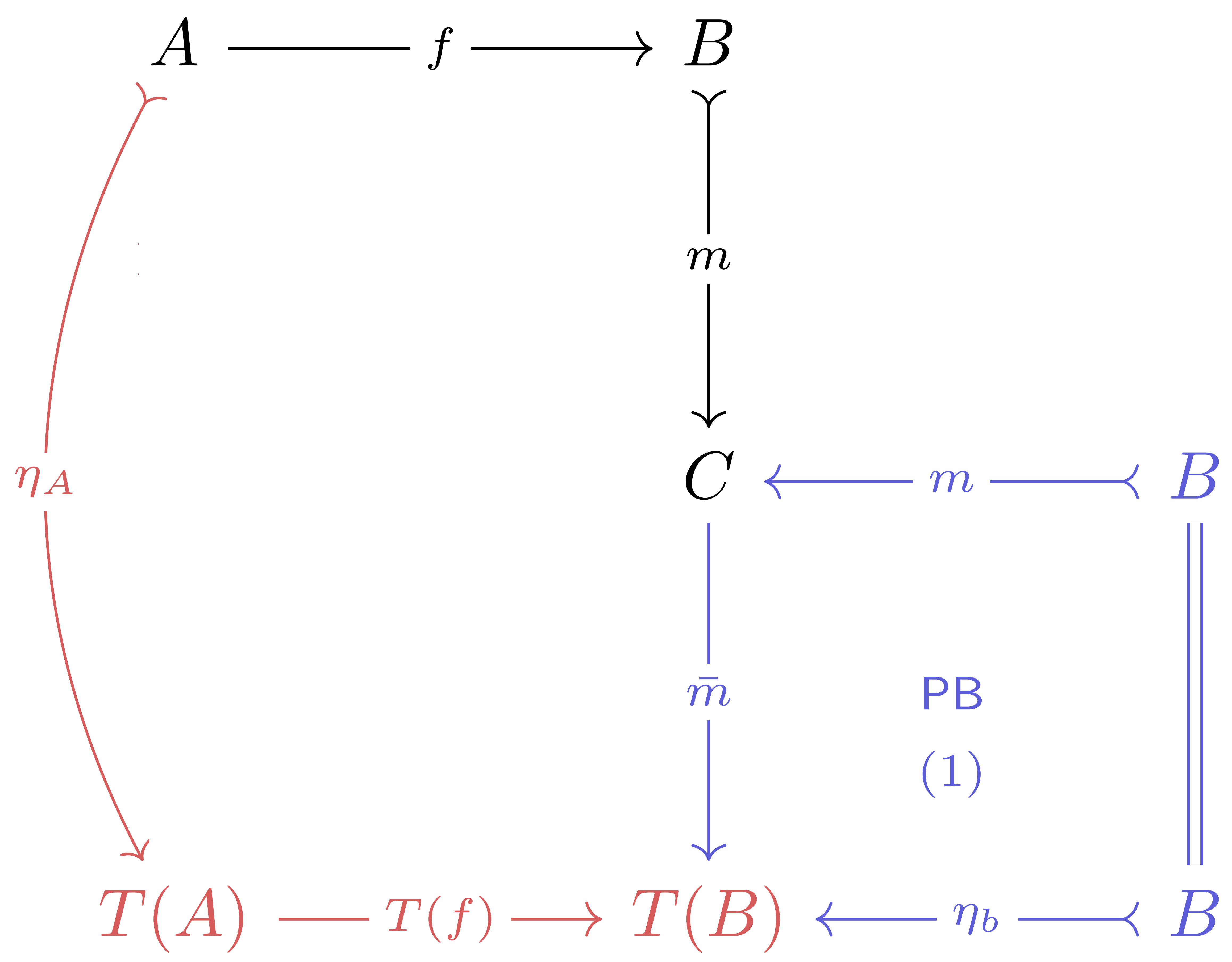

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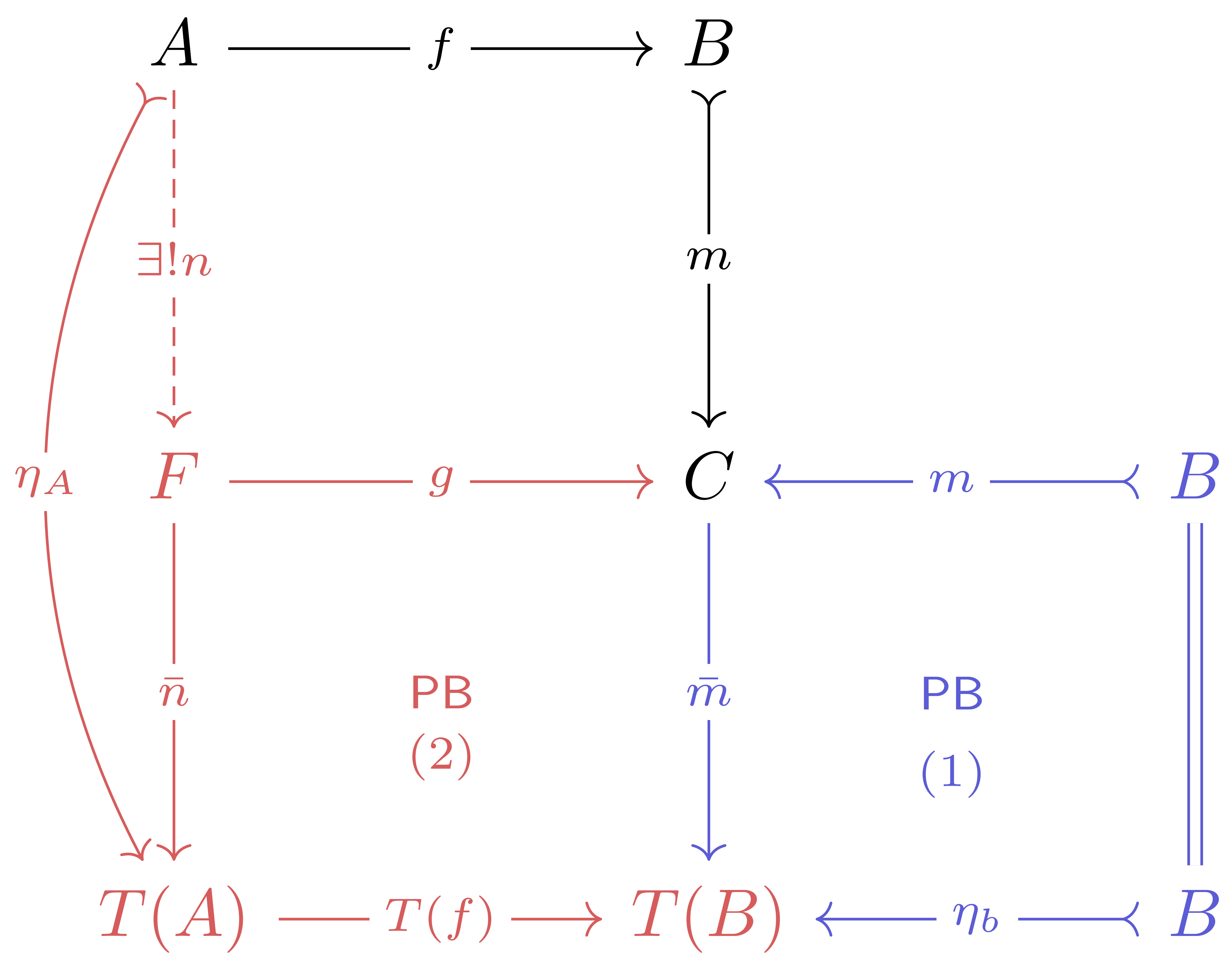
2. Construct $\mathsf{T(A)} \xleftarrow{\mathsf{n}} \mathsf{F} \xrightarrow{g} \mathsf{C}$ as the pullback of $\mathsf{T(A)} \xrightarrow{\mathsf{T(f)}} \mathsf{T(B)} \xleftarrow{\mathsf{m}} \mathsf{C}$ (cf. square $(2)$ below); by the universal property of pullbacks, this in addition entails the existence of a morphism $\mathsf{A} \xrightarrow{\mathsf{n}} \mathsf{F}$.



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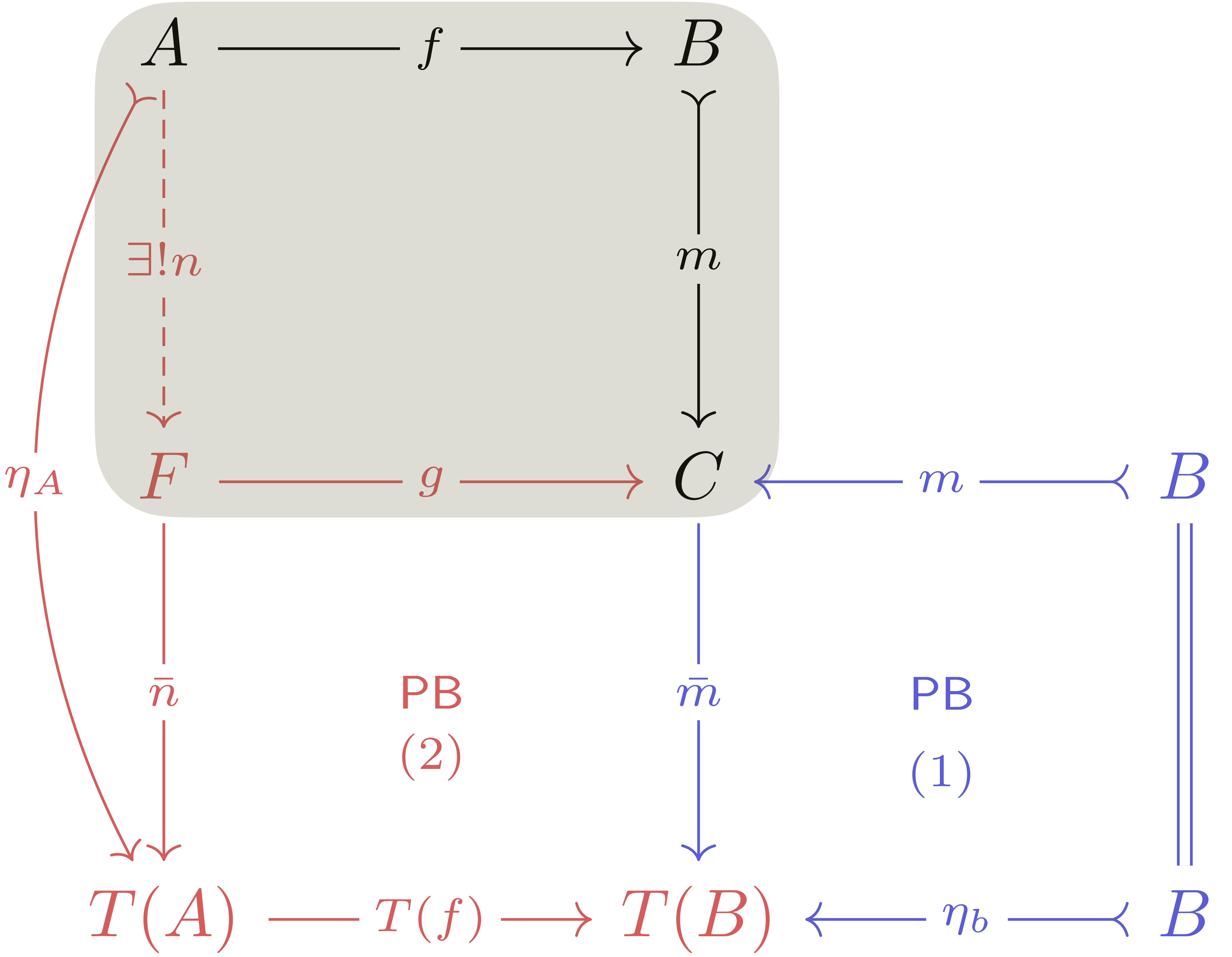
Then $(\mathsf{n}, \mathsf{g})$ is the FPC of $(\mathsf{f}, \mathsf{m})$, and $\mathsf{n}$ is in $\mathcal{M}$.



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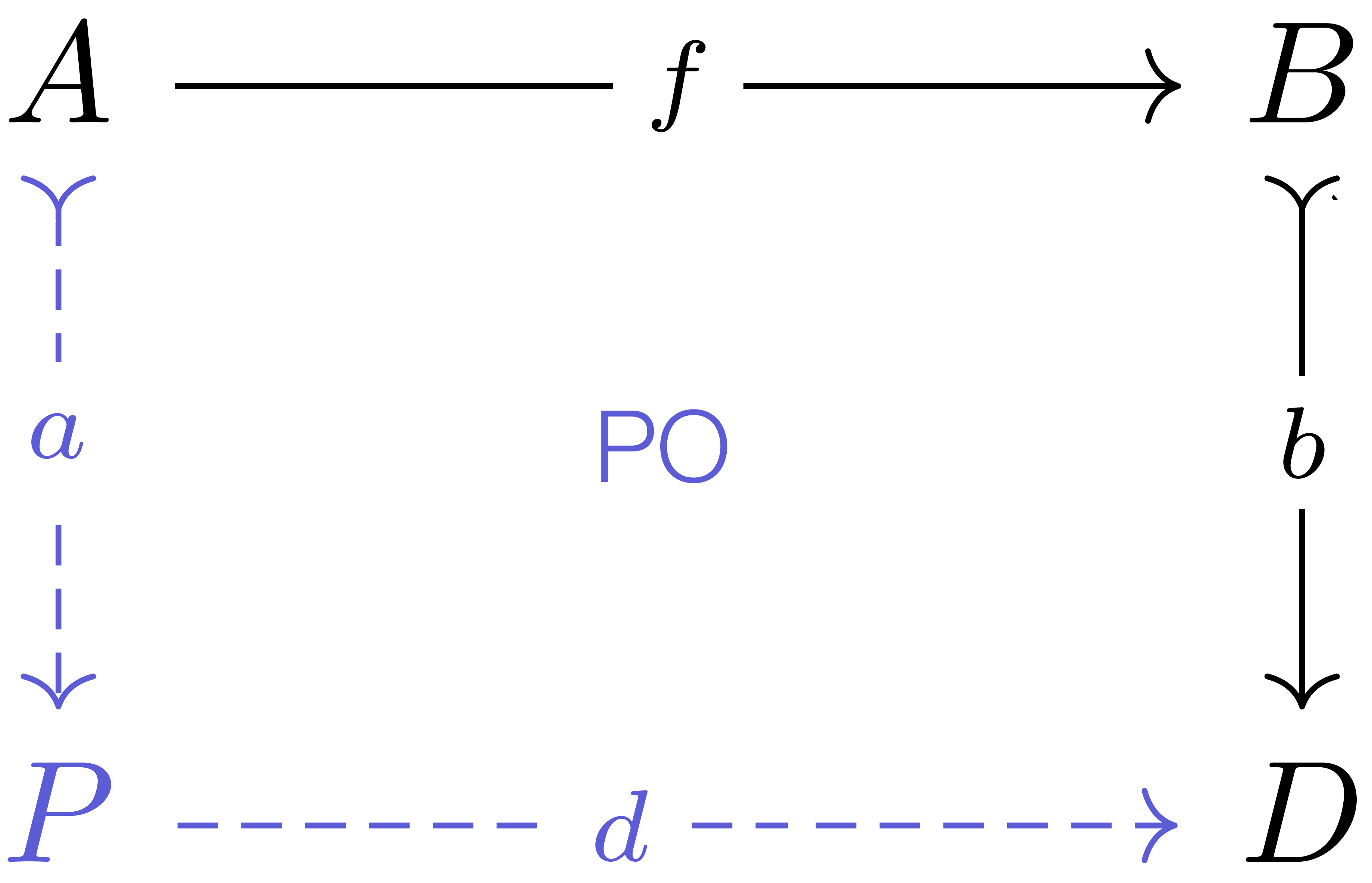
1. **Quasi-topoi** in rewriting theory
2. **Prerequisites** for non-linear rewriting
3. **Non-linear DPO rewriting**
4. ***Non-linear SqPO rewriting***
5. Conclusion and outlook

# (M-)multi-pushout-complements (MPOCs)

## Definition

For a category **C** with an $\mathcal{M}$**-partial map classifier**, the $\mathcal{M}$**-multi-pushout complement (mPOC)** $\mathcal{P}(\mathsf{f}, \mathsf{b})$ of a composable sequence of morphisms $\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{B}$ and $\mathsf{B} \xrightarrow{\mathsf{b}} \mathsf{D}$ with $\mathsf{b} \in \mathcal{M}$ is defined as

$$\mathcal{P}(f, b) := \{(A \xrightarrow{a} P, P \xrightarrow{d} D) \in \mathsf{mor}(\mathbf{C})^2 \mid a \in \mathcal{M} \wedge (d, b) = \mathsf{PO}(a, f)\}\,.$$



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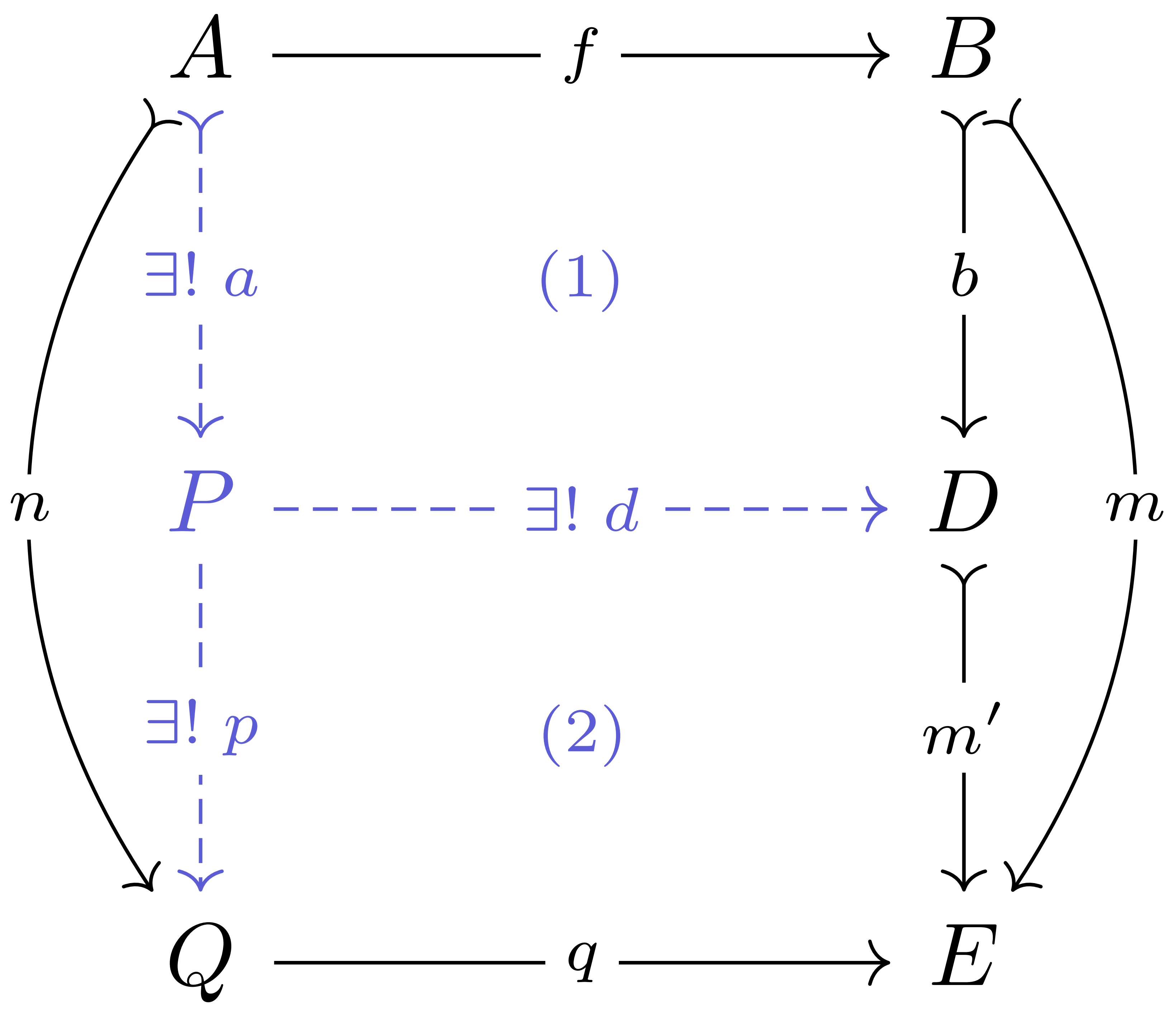
$$\mathcal{P}(f,b):=\{(A\xrightarrow{a}P,P\xrightarrow{d}D)\in\mathsf{mor}(\mathbf{C})^2\mid a\in\mathcal{M}\wedge(d,b)=\mathsf{PO}(a,f)\}\,.$$

## Proposition

In a quasi-topos **C** and for $\mathcal{M}=\mathsf{rm}(\mathbf{C})$ the class of regular monomorphisms, let $\mathcal{P}(\mathsf{f},\mathsf{b})$ be an mPOC.

- Universal property of $\mathcal{P}(\mathsf{f},\mathsf{b})$: for every diagram such as in (i) where $(1)+(2)$ is a pushout along an $\mathcal{M}$-morphism $\mathsf{n}$, and where $\mathsf{m}=\mathsf{m}'\circ\mathsf{b}$ for some $\mathsf{m}',\mathsf{b}\in\mathcal{M}$, there exists an element $(\mathsf{a},\mathsf{d})$ of $\mathcal{P}(\mathsf{f},\mathsf{b})$ and an $\mathcal{M}$-morphism $\mathsf{p}\in\mathcal{M}$ such that the diagram commutes and (2) is a pushout. Moreover, for any $\mathsf{p}'\in\mathcal{M}$ and for any other element $(\mathsf{a}',\mathsf{d}')$of $\mathcal{P}(\mathsf{f},\mathsf{b})$ with the same property, there exists an isomorphism $\delta\in\mathsf{iso}(\mathbf{C})$ such that $\delta\circ\mathsf{a}=\mathsf{a}'$ and $\mathsf{d}'\circ\delta=\mathsf{d}$.




(i)

# (M-)multi-pushout-complements (MPOCs)

## Definition

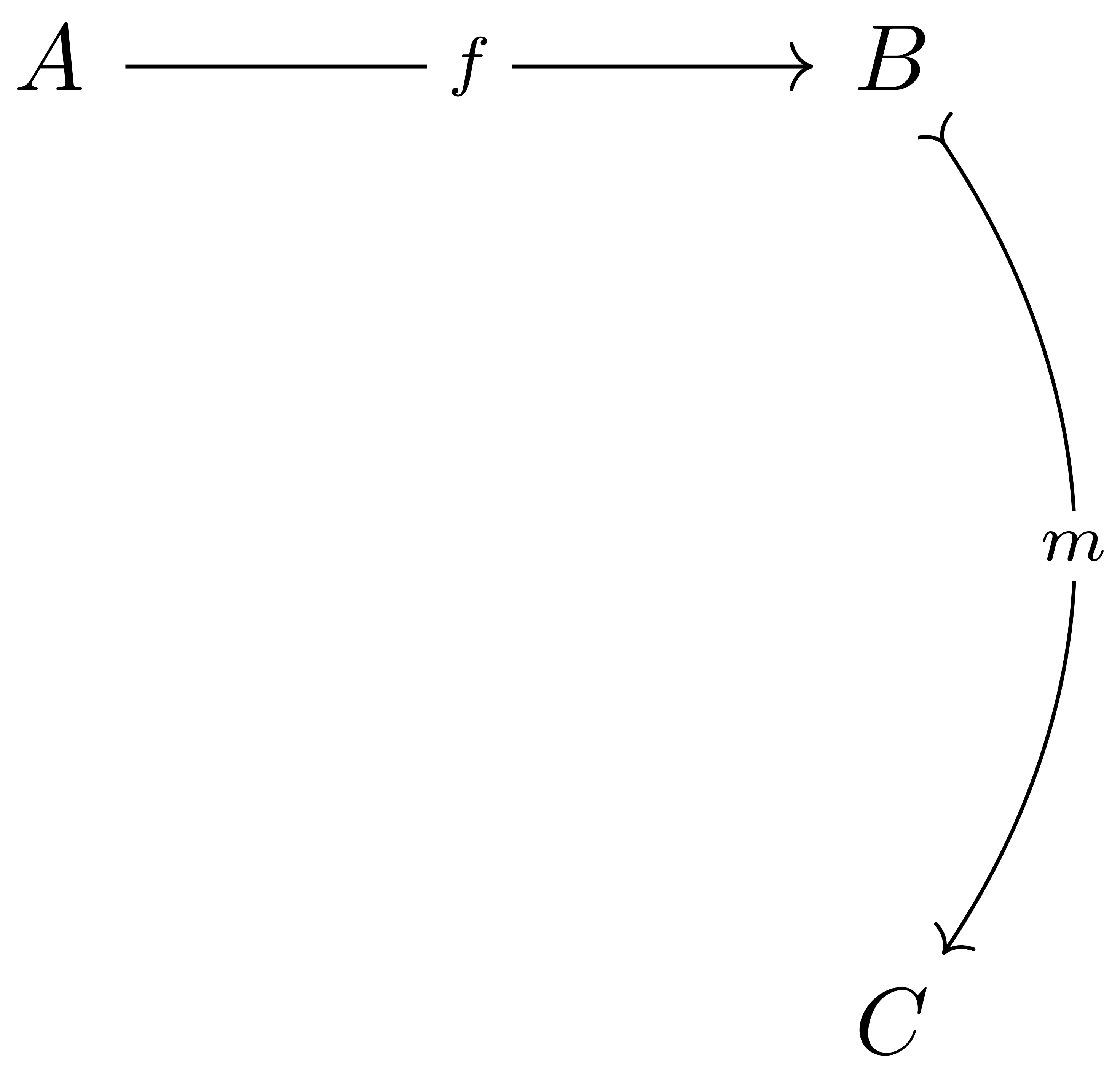
For a category **C** with an $\mathcal{M}$-partial map classifier, the $\mathcal{M}$-multi-pushout complement (mPOC) $\mathcal{P}(\mathsf{f},\mathsf{b})$ of a composable sequence of morphisms $\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{B}$ and $\mathsf{B} \xrightarrow{\mathsf{b}} \mathsf{D}$ with $\mathsf{b} \in \mathcal{M}$ is defined as

$$\mathcal{P}(f,b) := \{(A \xrightarrow{a} P, P \xrightarrow{d} D) \in \mathsf{mor}(\mathbf{C})^2 \mid a \in \mathcal{M} \wedge (d,b) = \mathsf{PO}(a,f)\}\,.$$

## Proposition

In a quasi-topos **C** and for $\mathcal{M} = \mathsf{rm}(\mathbf{C})$ the class of regular monomorphisms, let $\mathcal{P}(\mathsf{f},\mathsf{b})$ be an mPOC.

- Algorithm to compute $\mathcal{P}(\mathsf{f},\mathsf{b})$:




$(ii)$

# (M-)multi-pushout-complements (MPOCs)

## Definition

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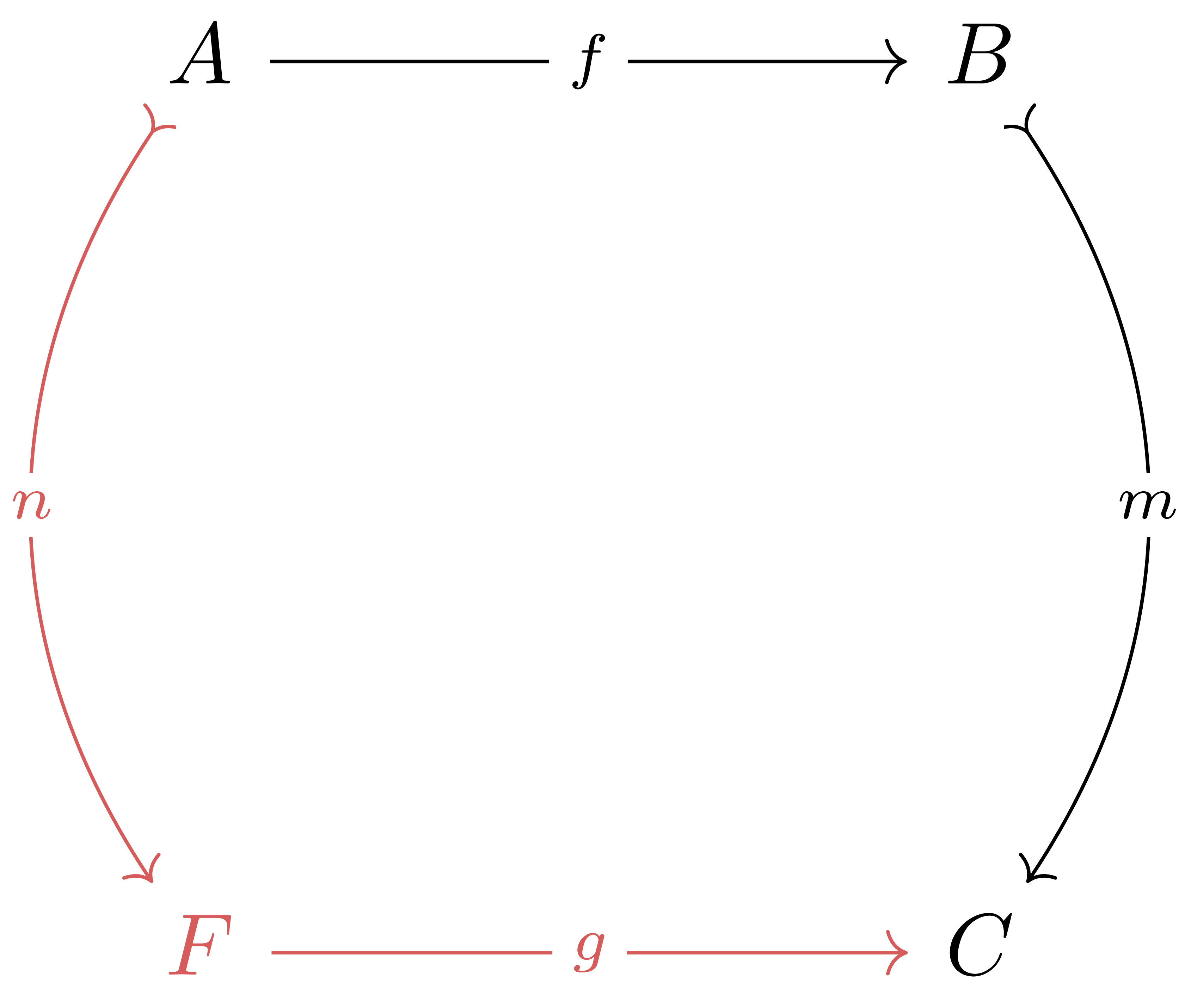
In a quasi-topos **C** and for $\mathcal{M} = \mathrm{rm}(\mathbf{C})$ the class of regular monomorphisms, let $\mathcal{P}(f, b)$ be an mPOC.

- Algorithm to compute $\mathcal{P}(f, b)$:

   1. Construct $(n, g)$ in diagram (ii) by taking the FPC of $(f, b)$.

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# (M-)multi-pushout-complements (MPOCs)

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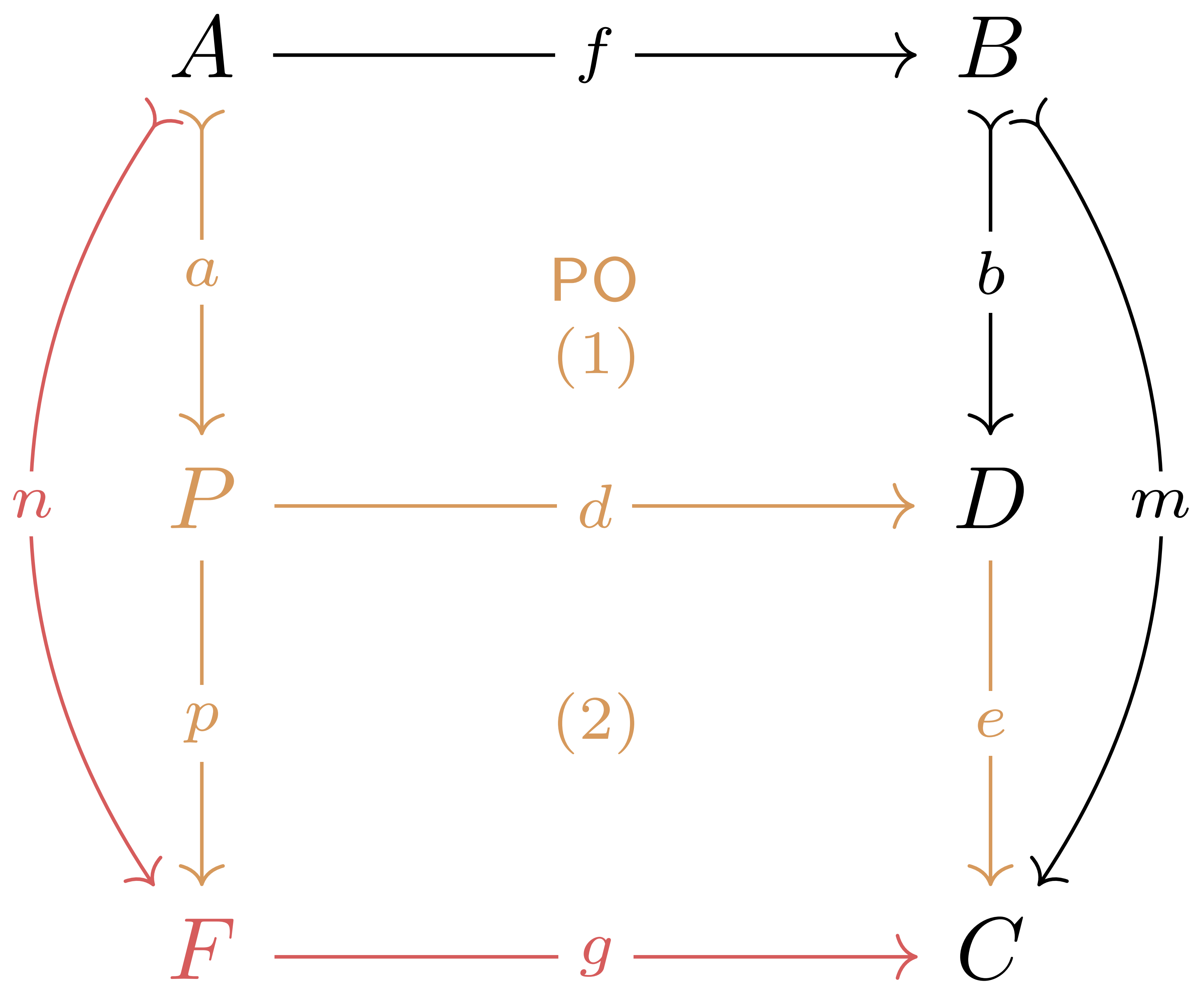
In a quasi-topos **C** and for $\mathcal{M} = \mathsf{rm}(\mathbf{C})$ the class of regular monomorphisms, let $\mathcal{P}(f, b)$ be an mPOC.

- Algorithm to compute $\mathcal{P}(f, b)$:
  1. Construct $(n, g)$ in diagram (ii) by taking the FPC of $(f, b)$.
  2. For every pair of morphisms $(a, p)$ such that $a \in \mathcal{M}$ and $a \circ p = n$, take the pushout $(1)$, which by universal property of pushouts induces an arrow $D \xrightarrow{e} C$; if $e \in \mathsf{iso}(\mathbf{C})$, $(a, d)$ is a contribution to the mPOC of $(f, b)$.




$(ii)$

# Final pullback complement augmentations (FPA)

## Definition

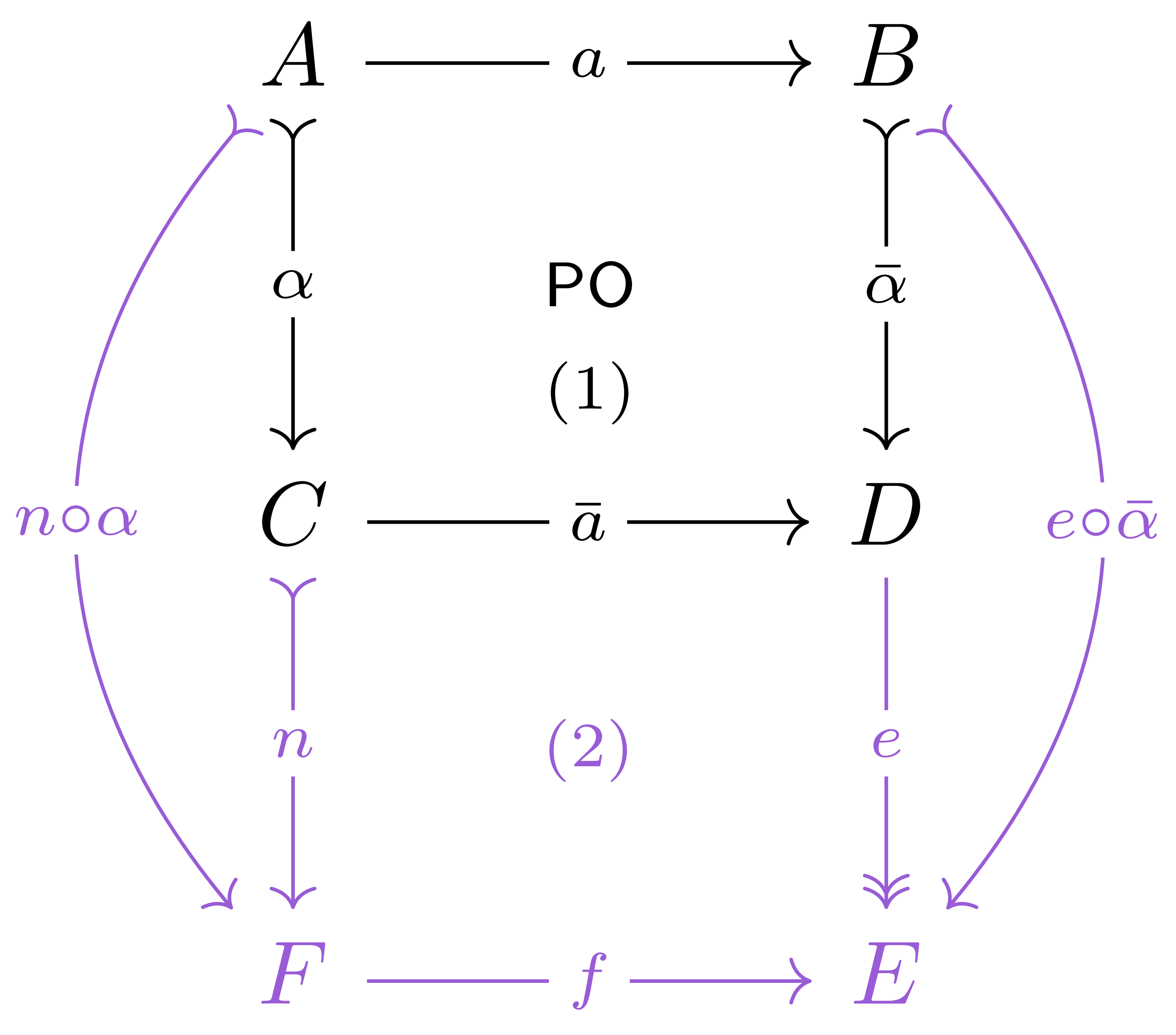
In a quasi-topos $\mathbf{C}$ with $\mathcal{M} = \mathsf{rm}(\mathbf{C})$, consider a pushout square along an $\mathcal{M}$-morphism such as square $(1)$ in the diagram on the right (where $\alpha, \bar{\alpha} \in \mathcal{M}$). We define an $\mathcal{M}$-FPC augmentation (FPA) of the pushout square $(1)$ as a diagram formed from an epimorphism $\mathsf{e} \in \mathsf{epi}(\mathbf{C})$ and that satisfies the following properties:

- The morphism $\mathsf{e} \circ \bar{\alpha}$ is an $\mathcal{M}$-morphism.
- $(\bar{\alpha}, \mathsf{id}_B)$ is a pullback of $(\mathsf{e}, \mathsf{e} \circ \bar{\alpha})$.
- Square $(1)+(2)$ is an FPC, and the induced morphism $\mathsf{n}$ that exists by the universal property of FPCs, here w.r.t. the FPC $(\mathsf{n} \circ \alpha, \mathsf{f})$ of $(\mathsf{a}, \mathsf{e} \circ \bar{\alpha})$, is an $\mathcal{M}$-morphism.

For a pushout as in $(1)$, we denote by $\mathsf{FPA}(\alpha, \mathsf{a})$ its class of FPAs:

$$\mathsf{FPA}(\alpha, a) := \{(n, f, e) \mid e \in \mathsf{epi}(\mathbf{C}) \wedge e \circ \bar{\alpha}, n \in \mathcal{M} \wedge (f, n \circ \alpha) = FPC(a, e \circ \bar{\alpha})\}$$

As induced by the properties of pushouts and of FPCs, FPAs are defined up to universal isomorphisms (in D, E and F), and for a given pushout square there will in general exist multiple non-isomorphic such augmentations.



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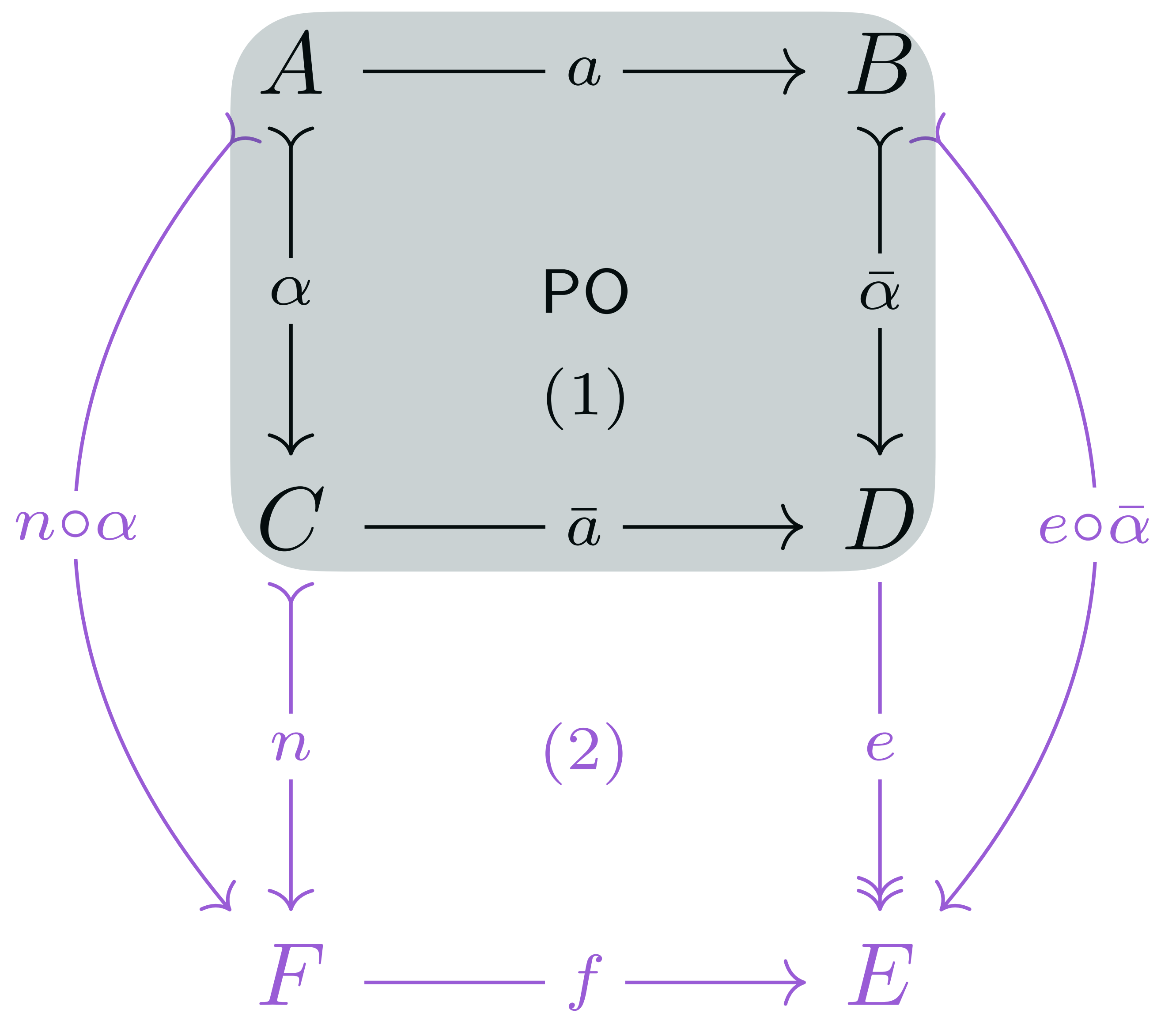

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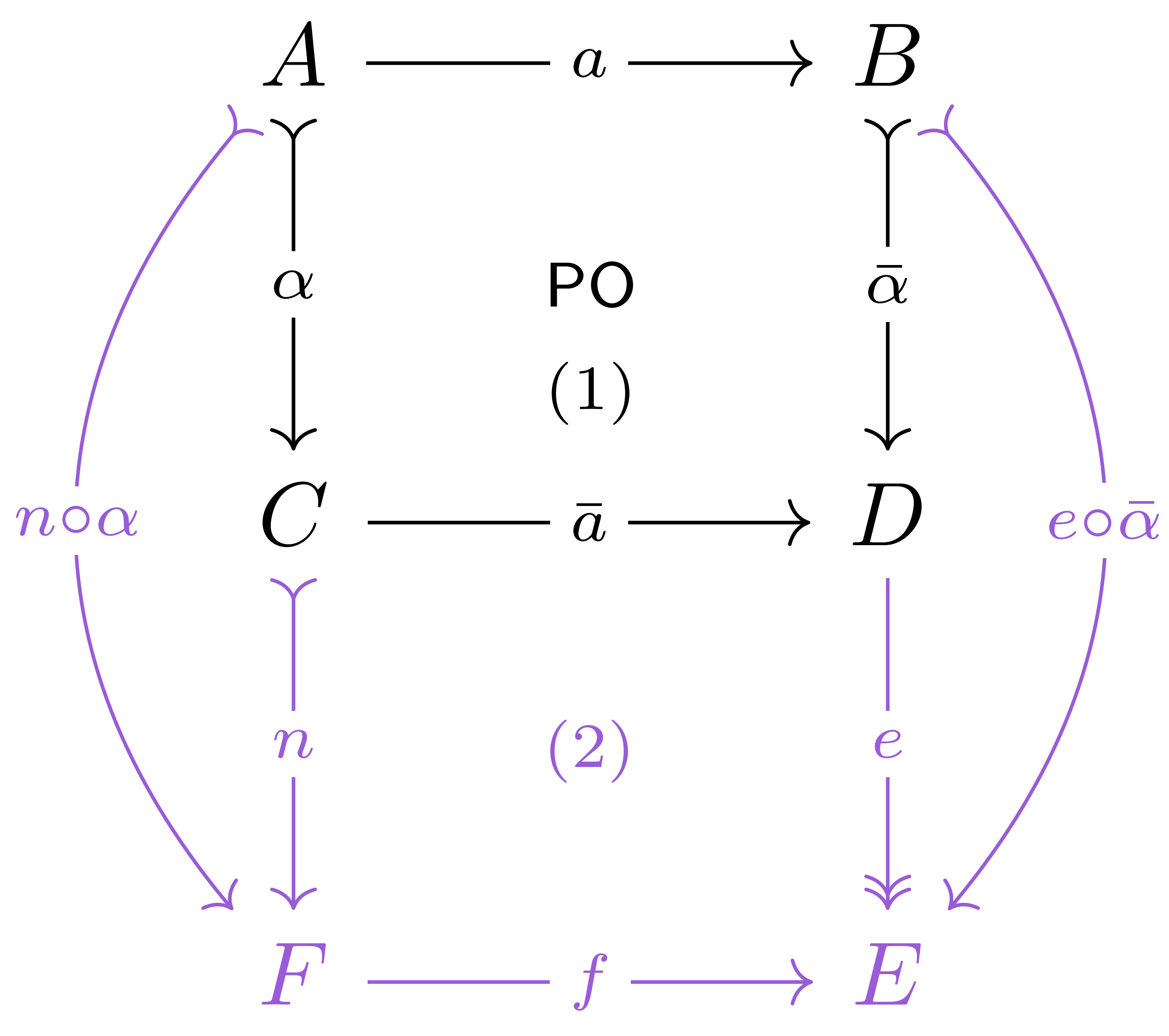


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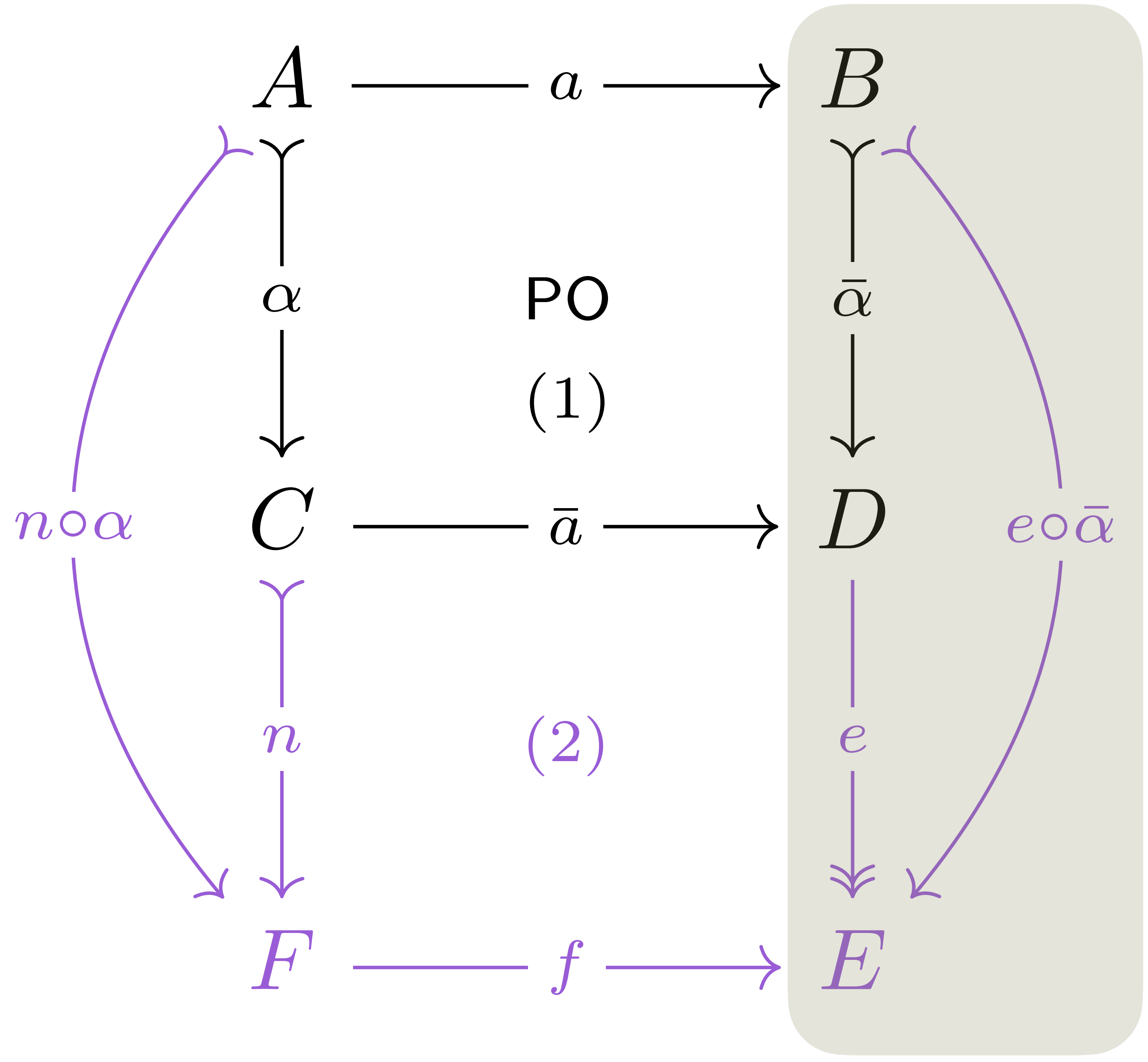


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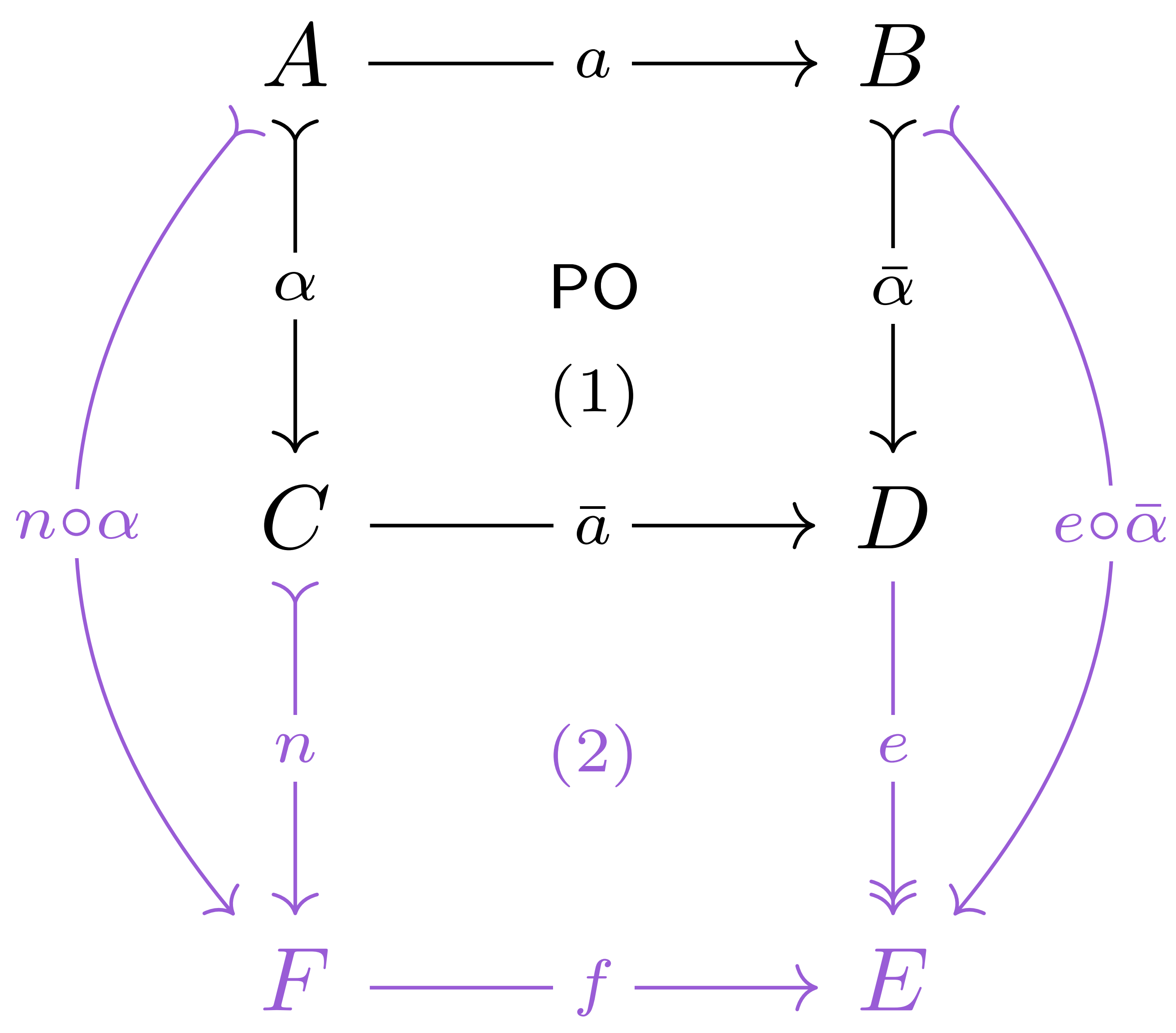

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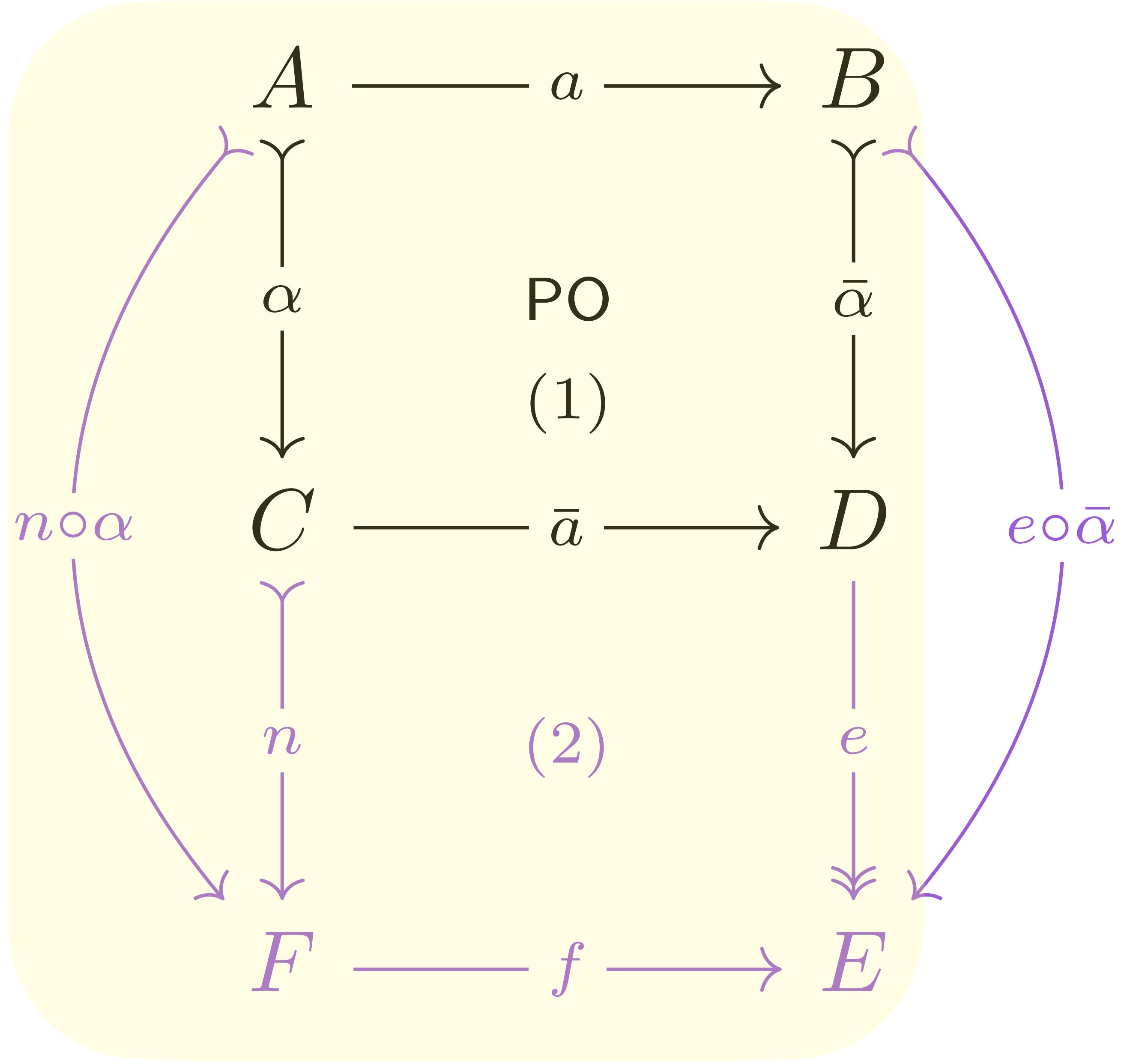

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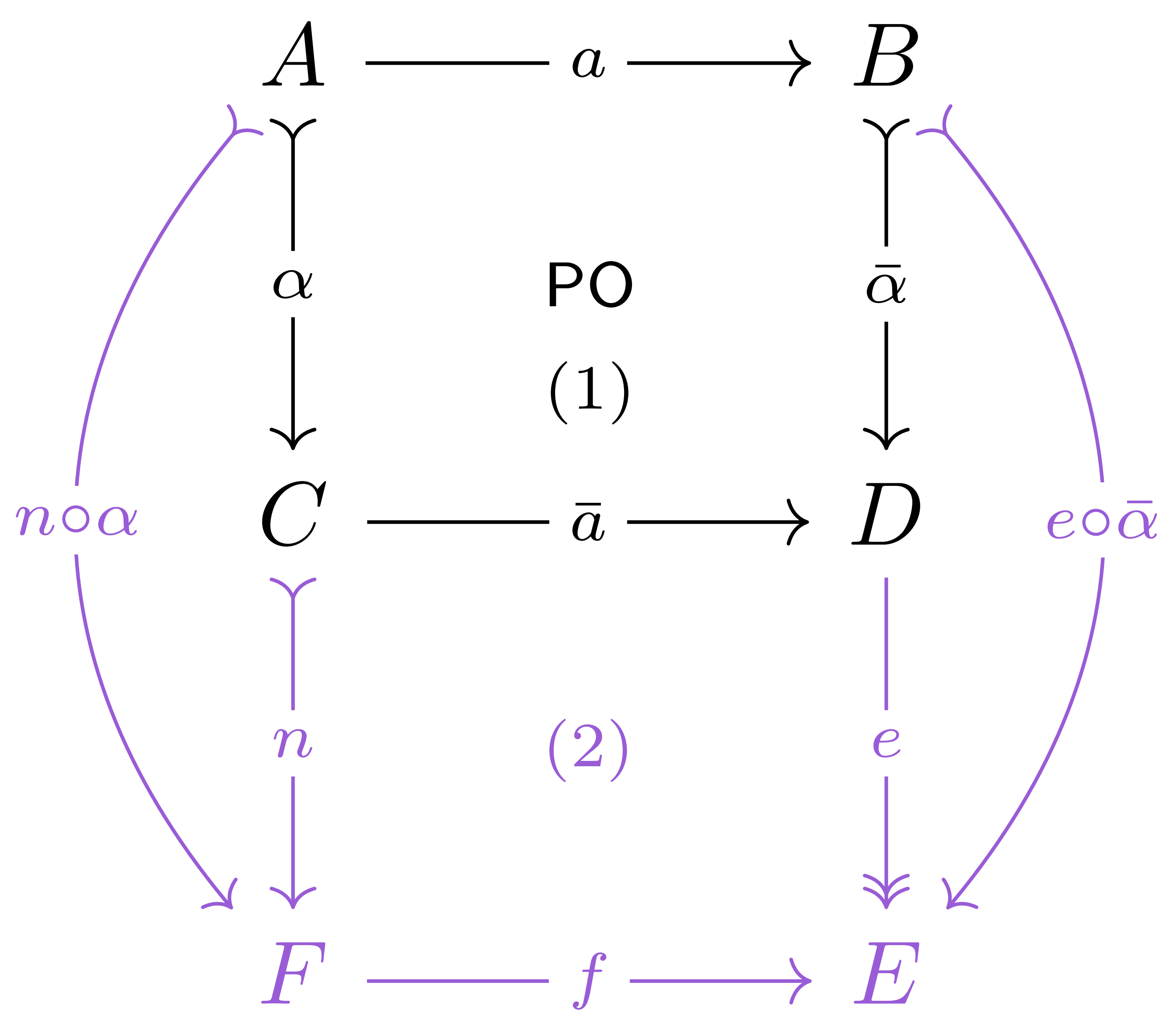
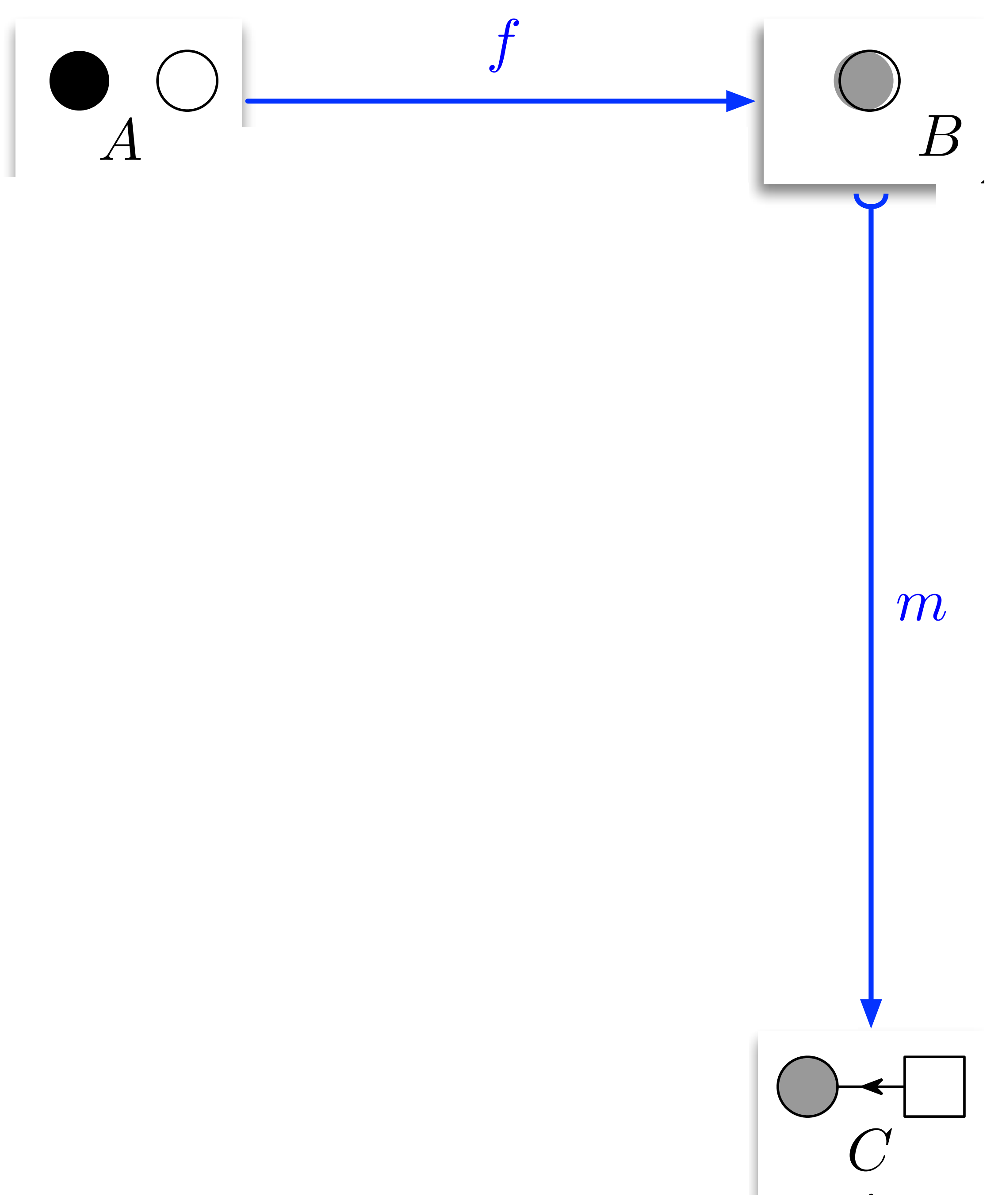


# **Example:** directed **simple** vs. directed **multi**-graphs



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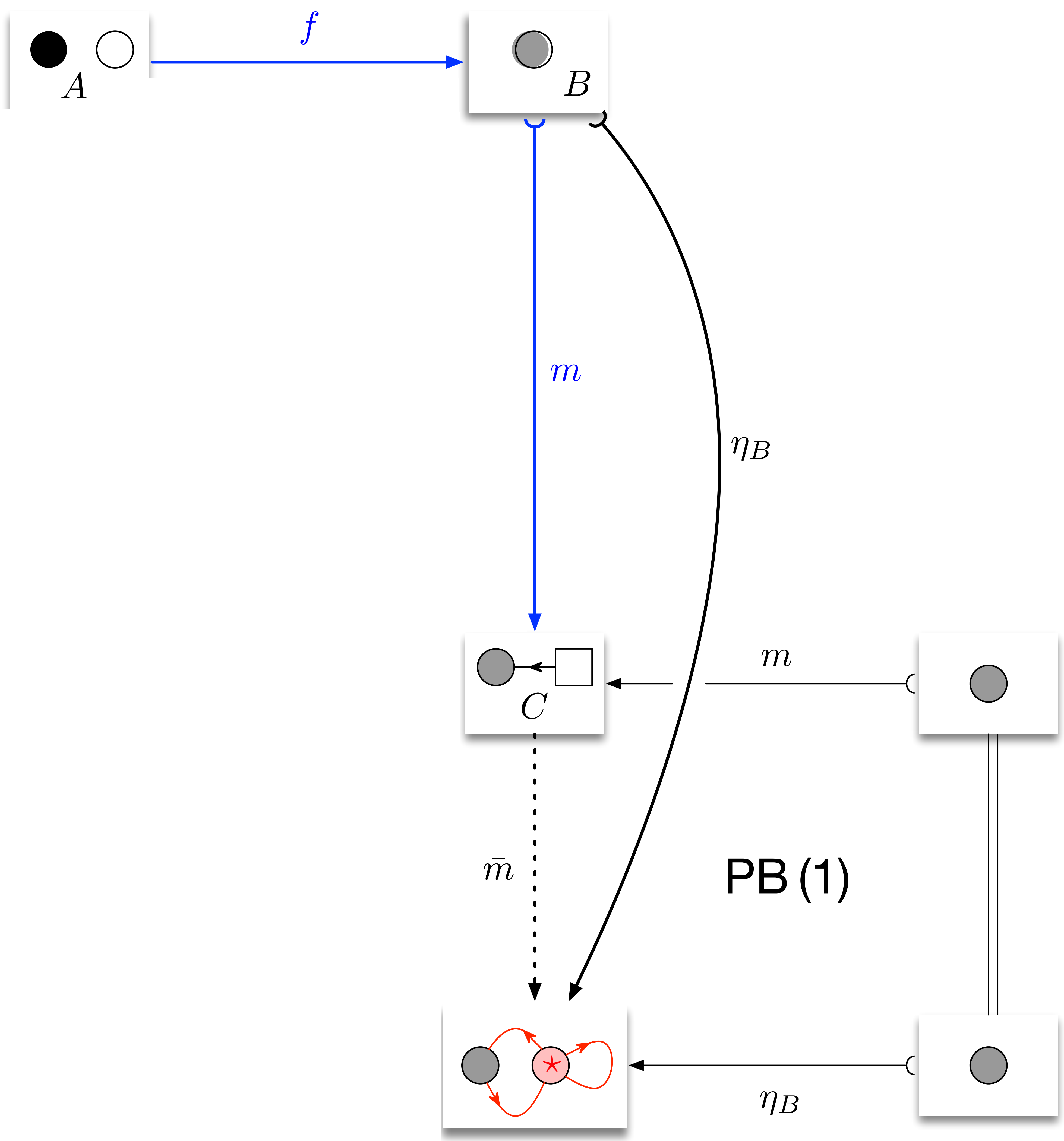

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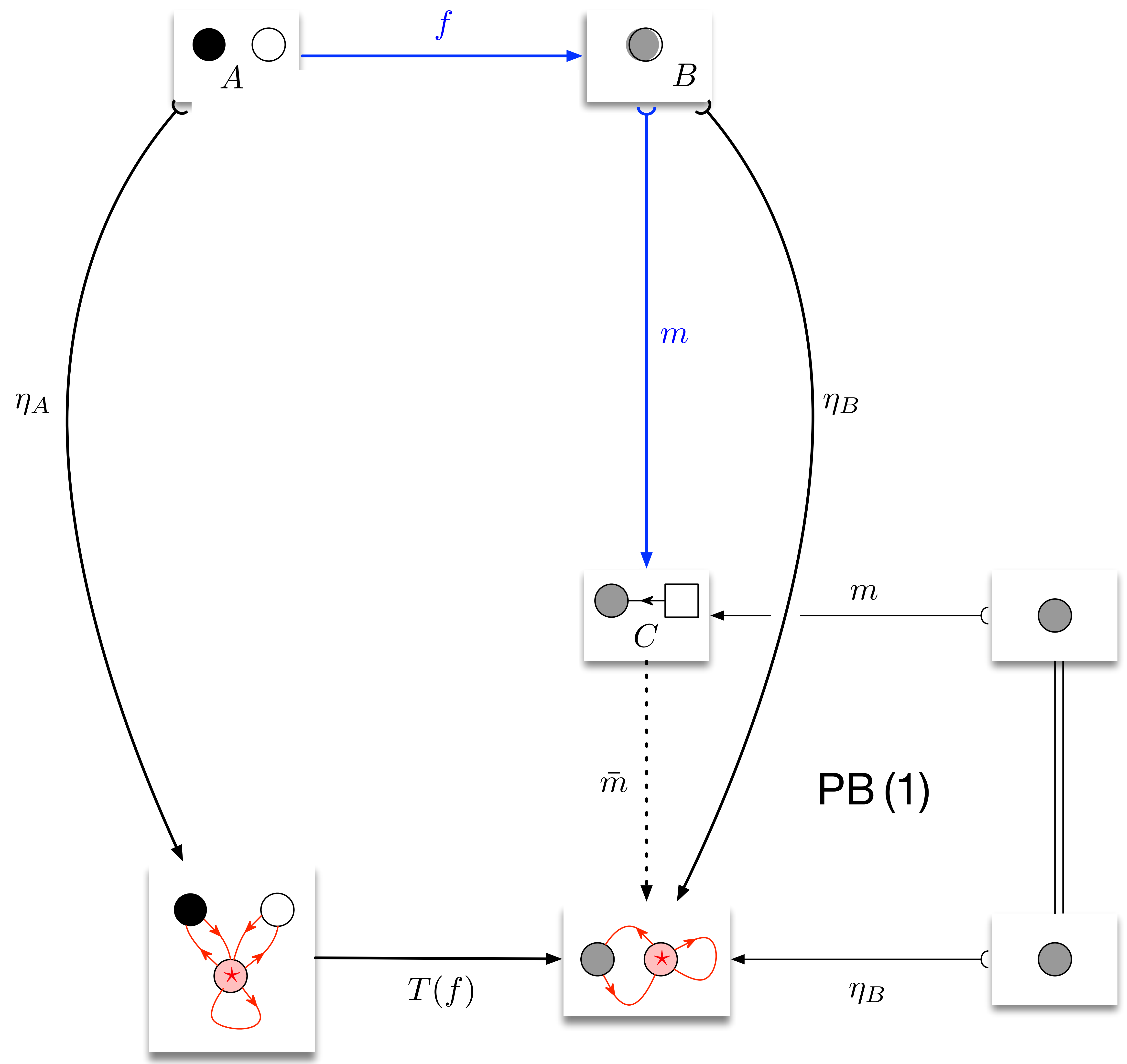

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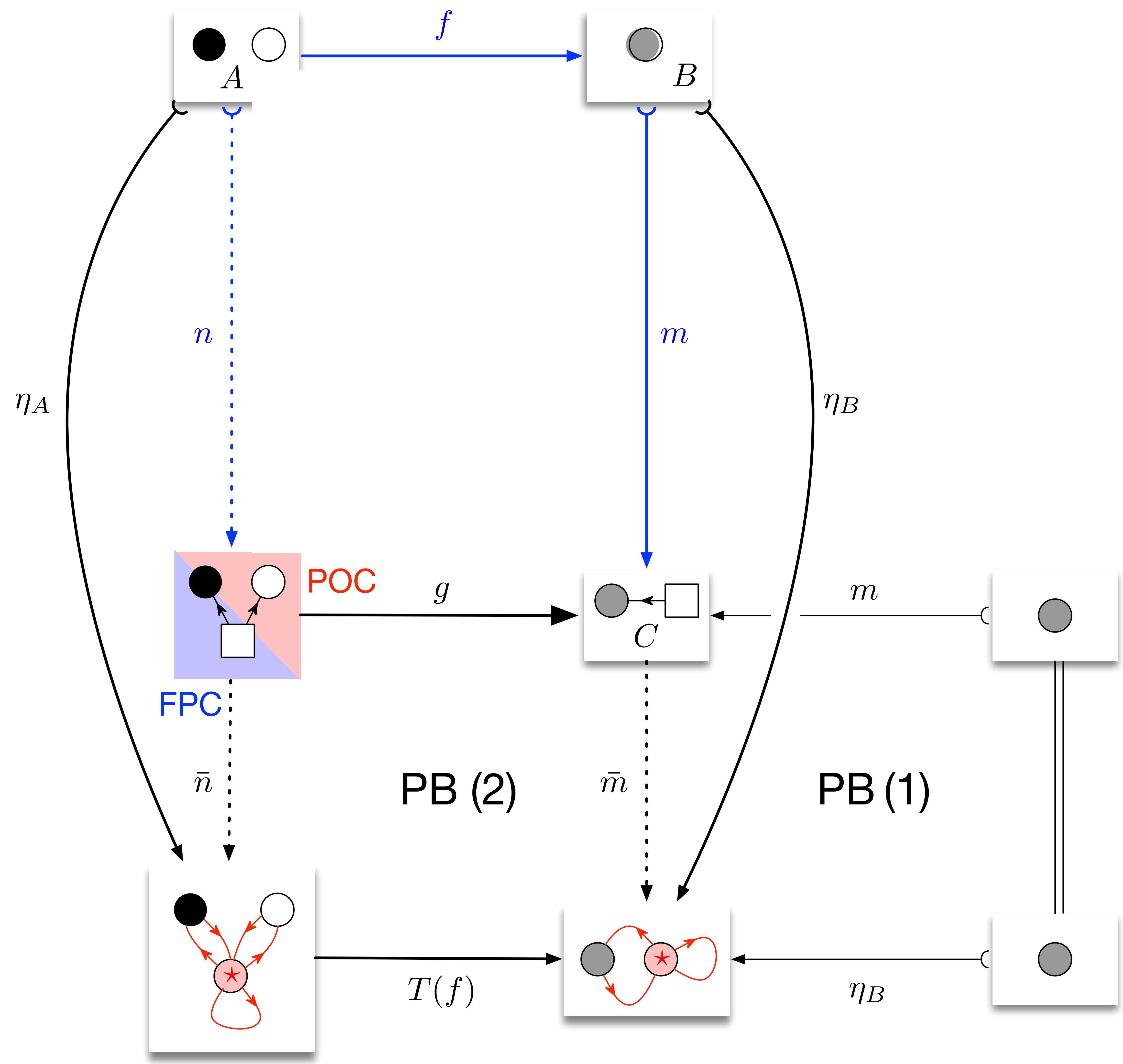

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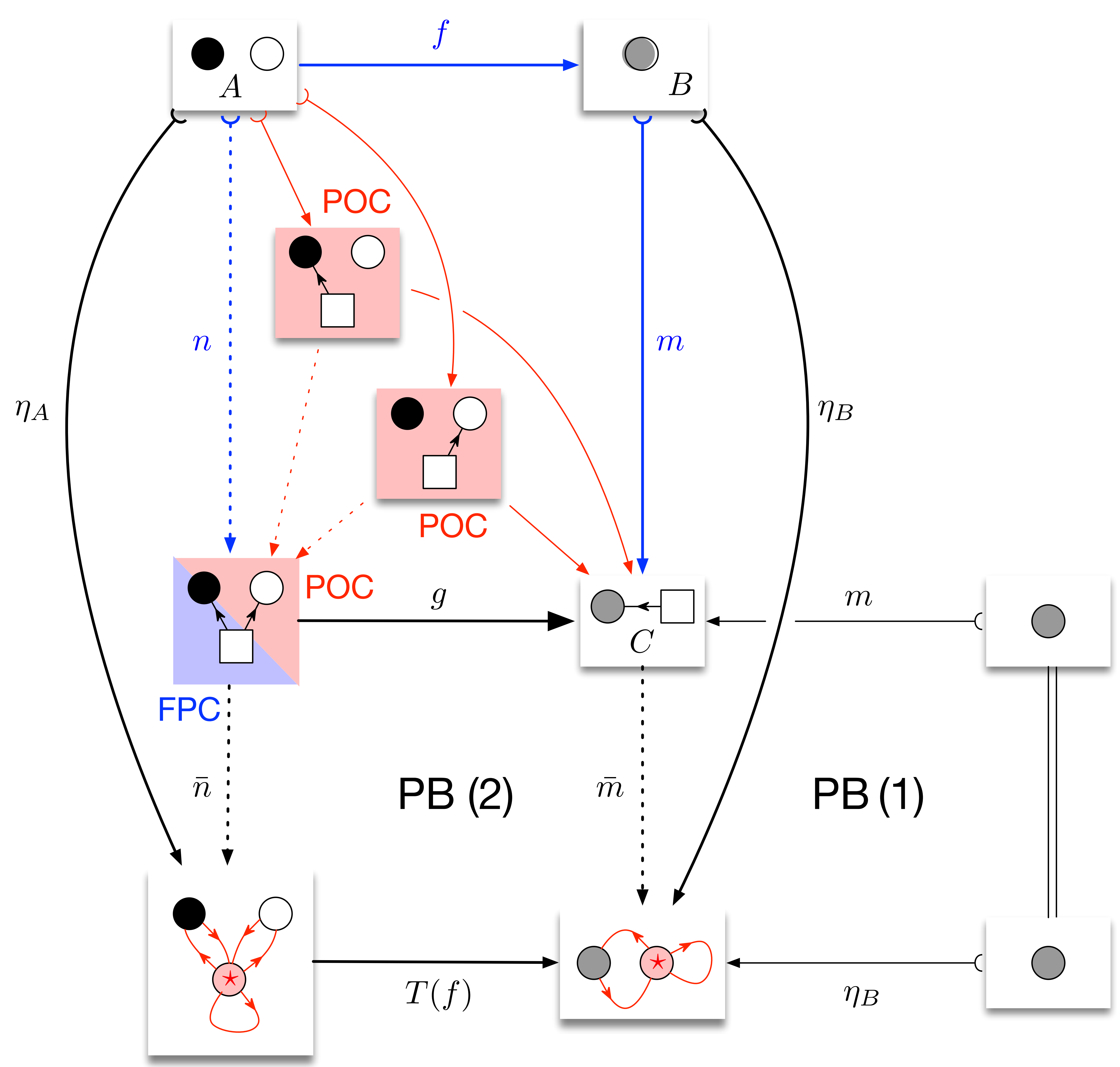

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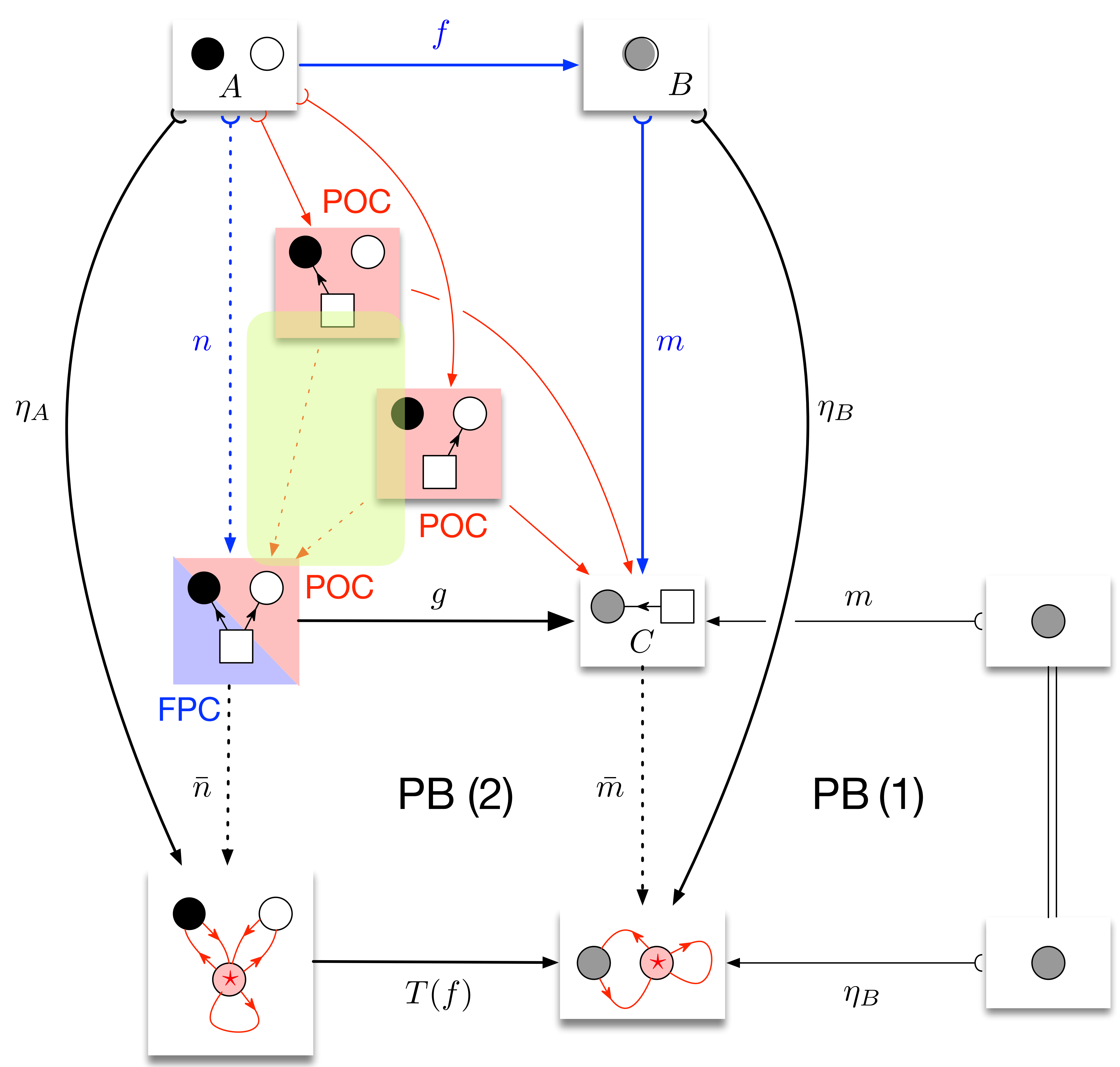

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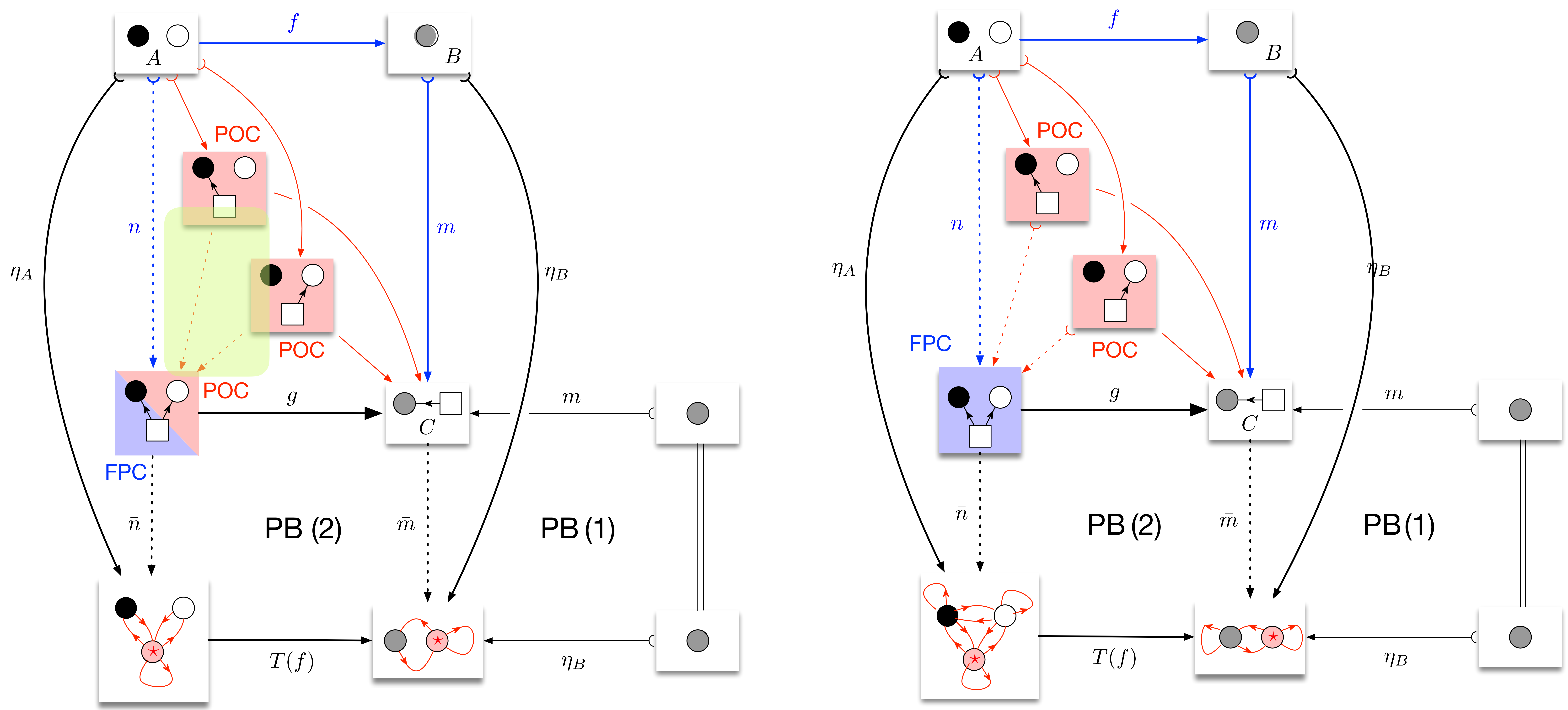
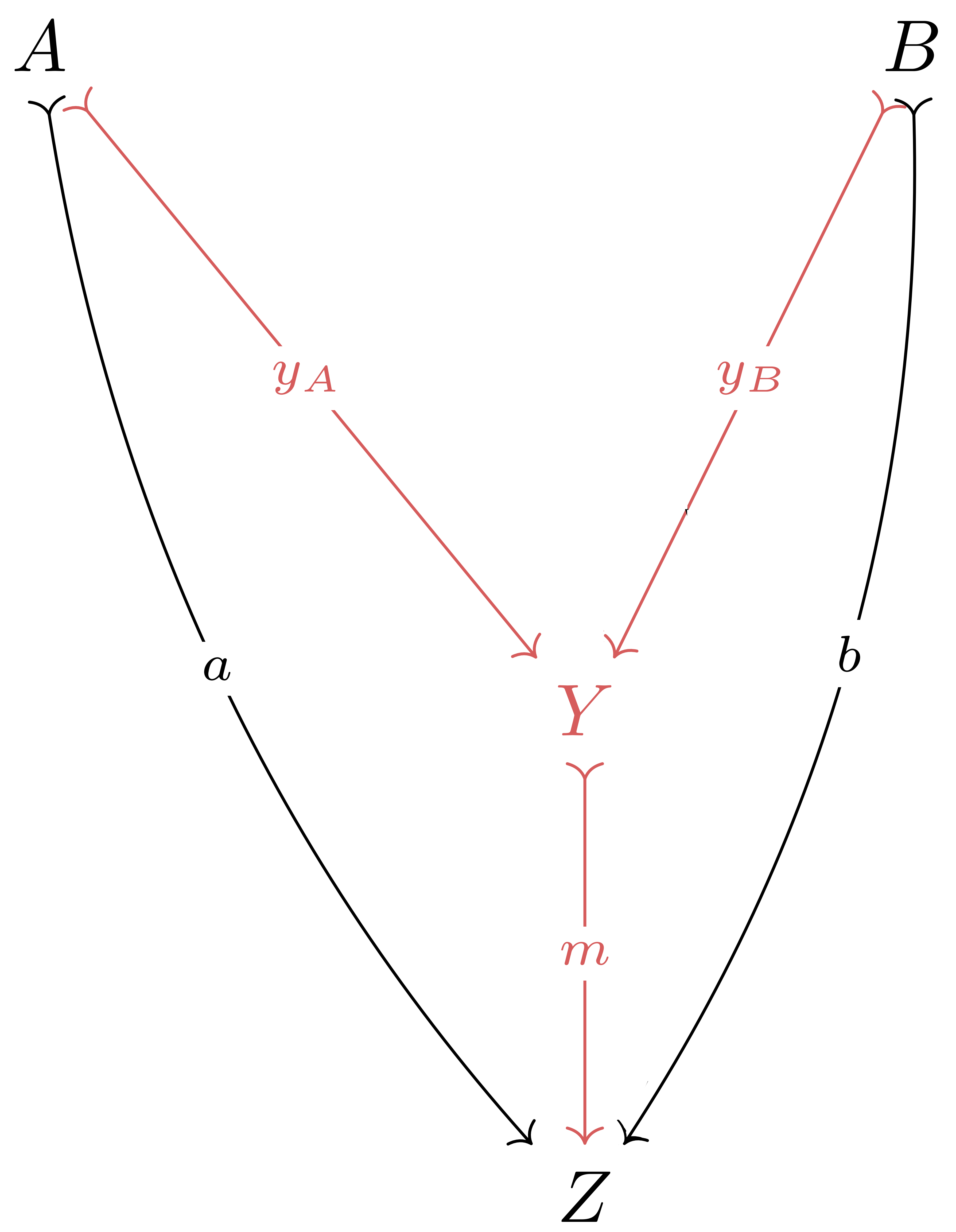


# Multi-sums

## Definition

In a quasi-topos $\mathbf{C}$, the multi-sum $\sum_{\mathcal{M}}(A, B)$ of two objects $A, B \in \mathsf{obj}(\mathbf{C})$ is defined as a family of cospans of regular monomorphisms $A \xrightarrow{y_A} Y \xleftarrow{y_B} B$ with the following universal property: for every cospan $A \xrightarrow{a} Z \xleftarrow{b} B$ with $a, b \in \mathsf{rm}(\mathbf{C})$, there exists an element $A \xrightarrow{y_A} Y \xleftarrow{y_B} B$ in $\sum_{\mathcal{M}}(A, B)$ and a regular monomorphism $Y \xrightarrow{m} Z$ such that $a = m \circ y_A$ and $b = m \circ y_B$, and moreover $(f, g)$ as well as $m$ are unique up to universal isomorphisms.

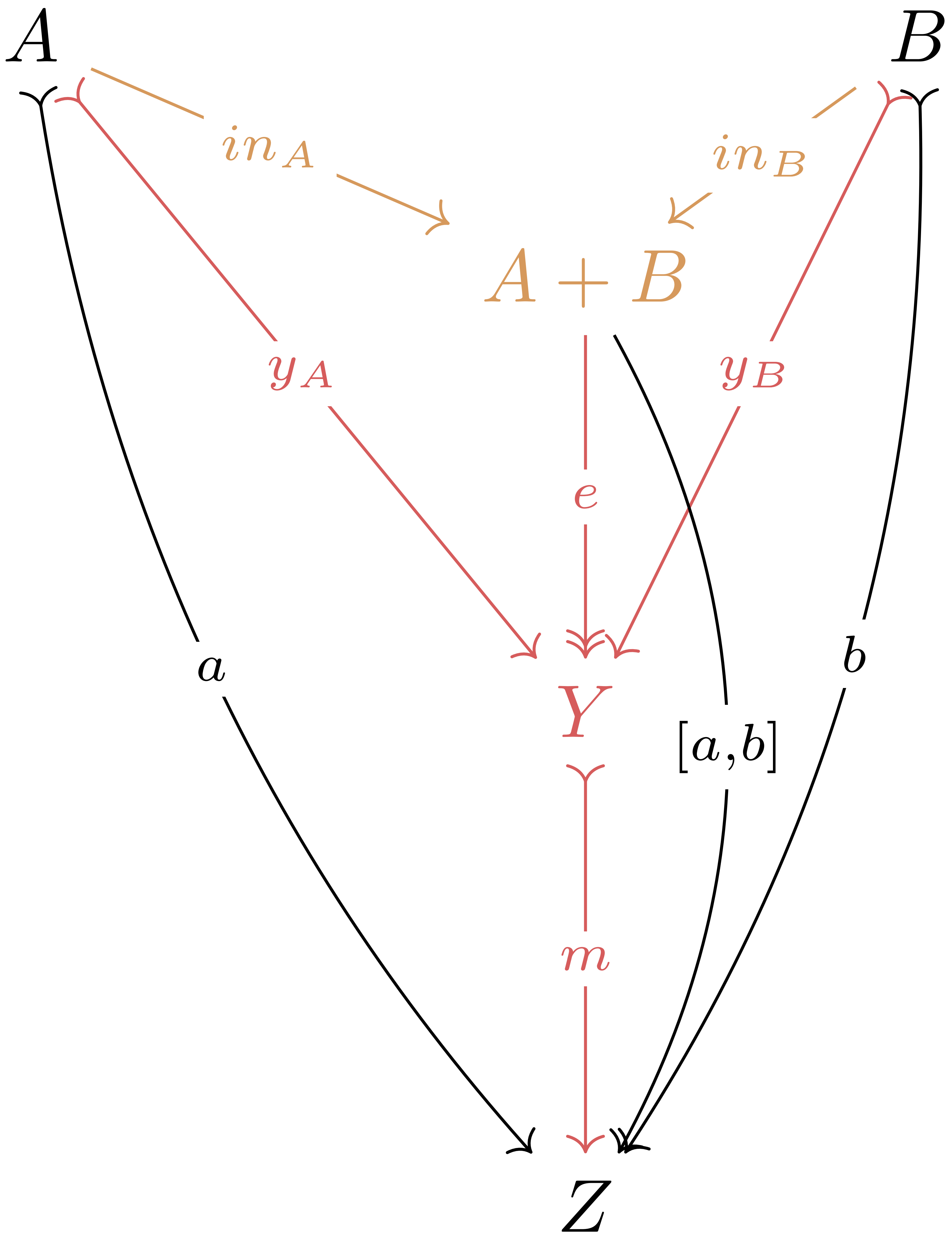


# Multi-sums

## Lemma

If $\mathsf{C}$ is a quasi-topos, the multi-sum $\sum_{\mathcal{M}}(\mathsf{A}, \mathsf{B})$ arises from the epi-$\mathcal{M}$-factorization of $\mathsf{C}$ (for $\mathcal{M} = \mathsf{rm}(\mathsf{C})$).

- **Existence:** Let $\mathsf{A} \xrightarrow{\mathsf{in_A}} \mathsf{A} + \mathsf{B} \xleftarrow{\mathsf{in_B}} \mathsf{B}$ be the disjoint union of A and B. Then for any cospan $\mathsf{A} \xrightarrow{\mathsf{a}} \mathsf{Z} \xleftarrow{\mathsf{b}} \mathsf{B}$ with $\mathsf{a}, \mathsf{b} \in \mathcal{M}$, the epi-$\mathcal{M}$-factorization of the induced arrow $\mathsf{A} + \mathsf{B} \xrightarrow{[\mathsf{a},\mathsf{b}]} \mathsf{Z}$ into an epimorphism $\mathsf{A} + \mathsf{B} \xrightarrow{\mathsf{e}} \mathsf{Y}$ and an $\mathcal{M}$-morphism $\mathsf{Y} \xrightarrow{\mathsf{m}} \mathsf{Z}$ yields a cospan $(\mathsf{y_A} = \mathsf{e} \circ \mathsf{in_A}, \mathsf{y_B} = \mathsf{e} \circ \mathsf{in_B})$, which by the decomposition property of $\mathcal{M}$-morphisms is a cospan of $\mathcal{M}$-morphisms.



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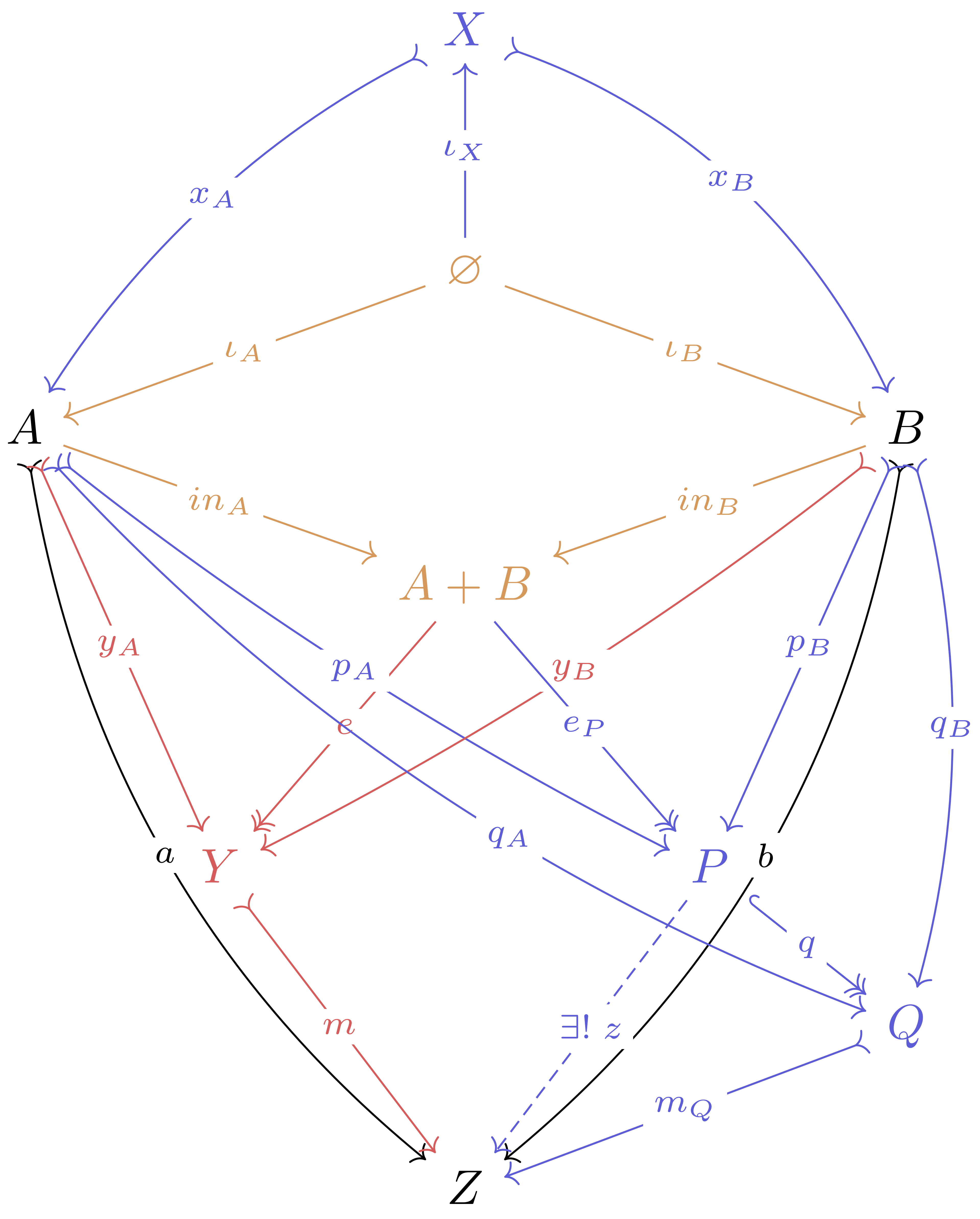
- **Construction:** For objects $A, B \in \mathsf{obj}(\mathbf{C})$, every element $A \xrightarrow{q_A} Q \xleftarrow{q_B} B$ in $\sum_{\mathcal{M}}(A, B)$ is obtained from a pushout of some span $A \xleftarrow{x_A} X \xrightarrow{x_B} B$ with $x_A, x_B \in \mathcal{M}$ and a morphism $P \xrightarrow{q} Q$ in $\mathsf{mono}(\mathbf{C}) \cap \mathsf{epi}(\mathbf{C})$.



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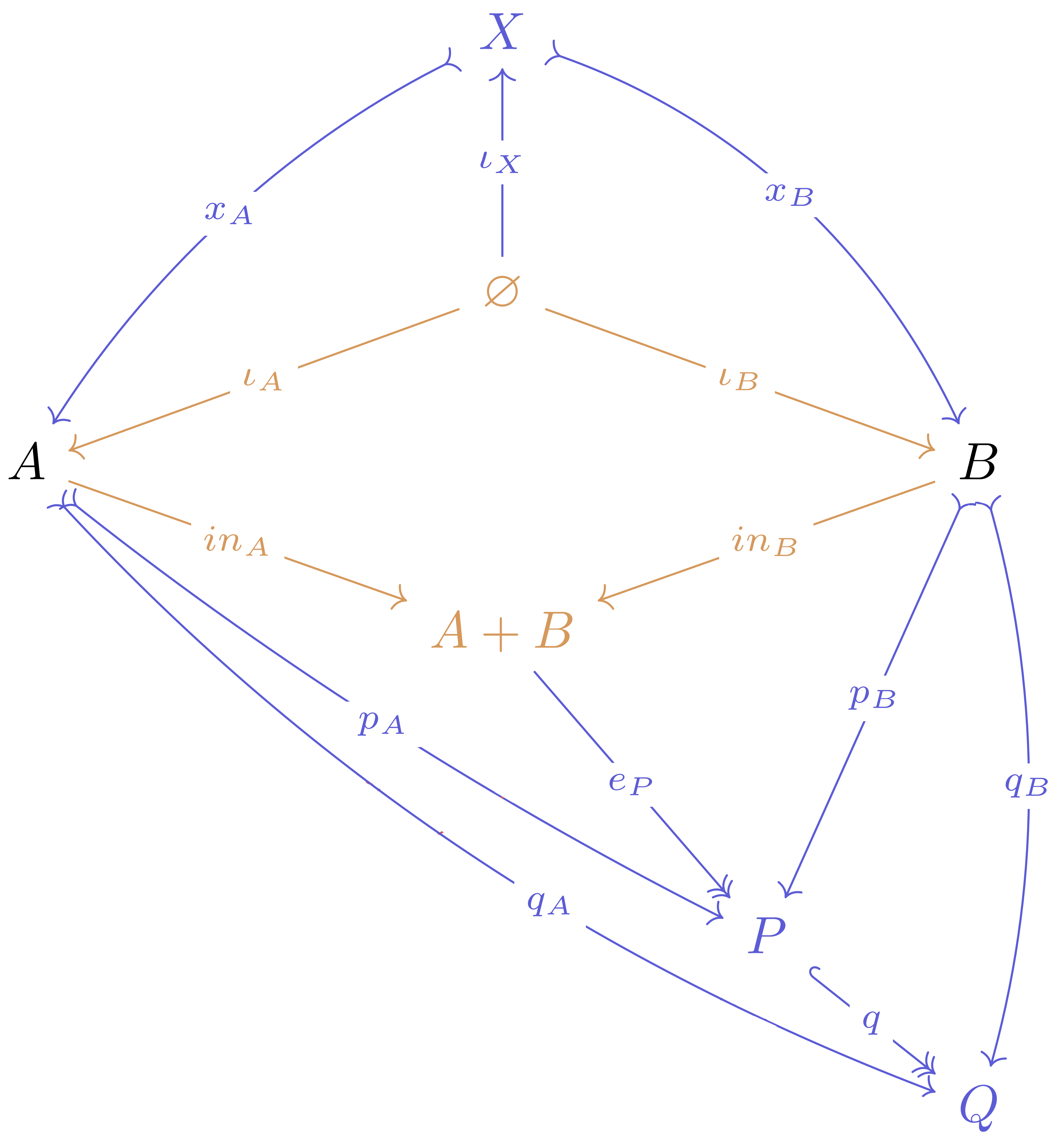
- **Construction:** For objects $\mathsf{A},\mathsf{B} \in \mathsf{obj}(\mathsf{C})$, every element $\mathsf{A} \xrightarrow{\mathsf{q_A}} \mathsf{Q} \xleftarrow{\mathsf{q_B}} \mathsf{B}$ in $\sum_{\mathcal{M}}(\mathsf{A},\mathsf{B})$ is obtained from a pushout of some span $\mathsf{A} \xleftarrow{\mathsf{x_A}} \mathsf{X} \xrightarrow{\mathsf{x_B}} \mathsf{B}$ with $\mathsf{x_A},\mathsf{x_B} \in \mathcal{M}$ and a morphism $\mathsf{P} \xrightarrow{\mathsf{q}} \mathsf{Q}$ in $\mathsf{mono}(\mathsf{C}) \cap \mathsf{epi}(\mathsf{C})$.

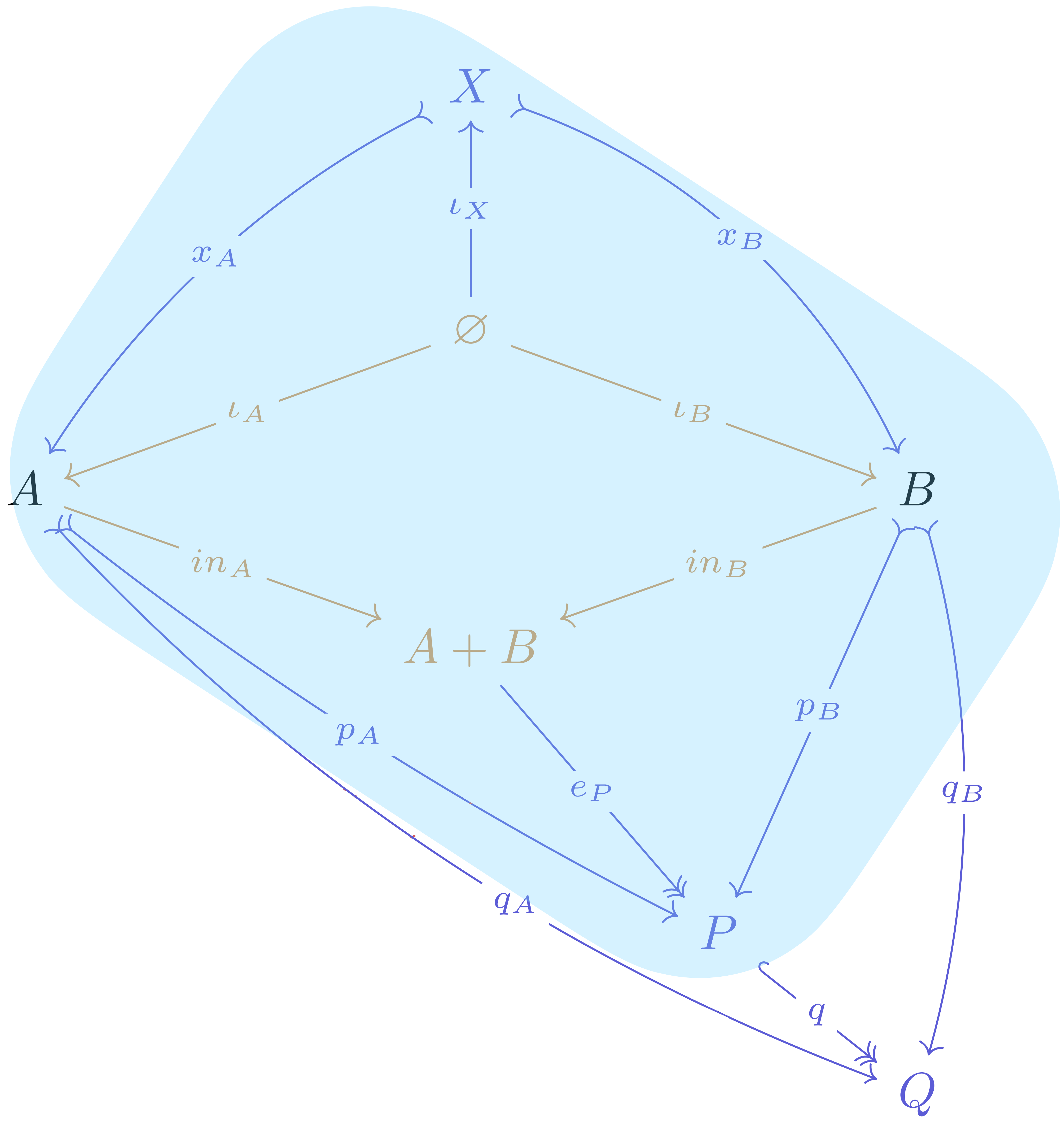


# Multi-sums

## Lemma

If $\mathbf{C}$ is a quasi-topos, the multi-sum $\sum_{\mathcal{M}}(A, B)$ arises from the epi-$\mathcal{M}$-factorization of $\mathbf{C}$ (for $\mathcal{M} = \mathsf{rm}(\mathbf{C})$).

- **Existence:** Let $A \xrightarrow{in_A} A + B \xleftarrow{in_B} B$ be the disjoint union of A and B. Then for any cospan $A \xrightarrow{a} Z \xleftarrow{b} B$ with $a, b \in \mathcal{M}$, the epi-$\mathcal{M}$-factorization of the induced arrow $A + B \xrightarrow{[a,b]} Z$ into an epimorphism $A + B \xrightarrow{e} Y$ and an $\mathcal{M}$-morphism $Y \xrightarrow{m} Z$ yields a cospan $(y_A = e \circ in_A, y_B = e \circ in_B)$, which by the decomposition property of $\mathcal{M}$-morphisms is a cospan of $\mathcal{M}$-morphisms.
- **Construction:** For objects $A, B \in \mathsf{obj}(\mathbf{C})$, every element $A \xrightarrow{q_A} Q \xleftarrow{q_B} B$ in $\sum_{\mathcal{M}}(A, B)$ is obtained from a pushout of some span $A \xleftarrow{x_A} X \xrightarrow{x_B} B$ with $x_A, x_B \in \mathcal{M}$ and a morphism $P \xrightarrow{q} Q$ in $\mathsf{mono}(\mathbf{C}) \cap \mathsf{epi}(\mathbf{C})$.

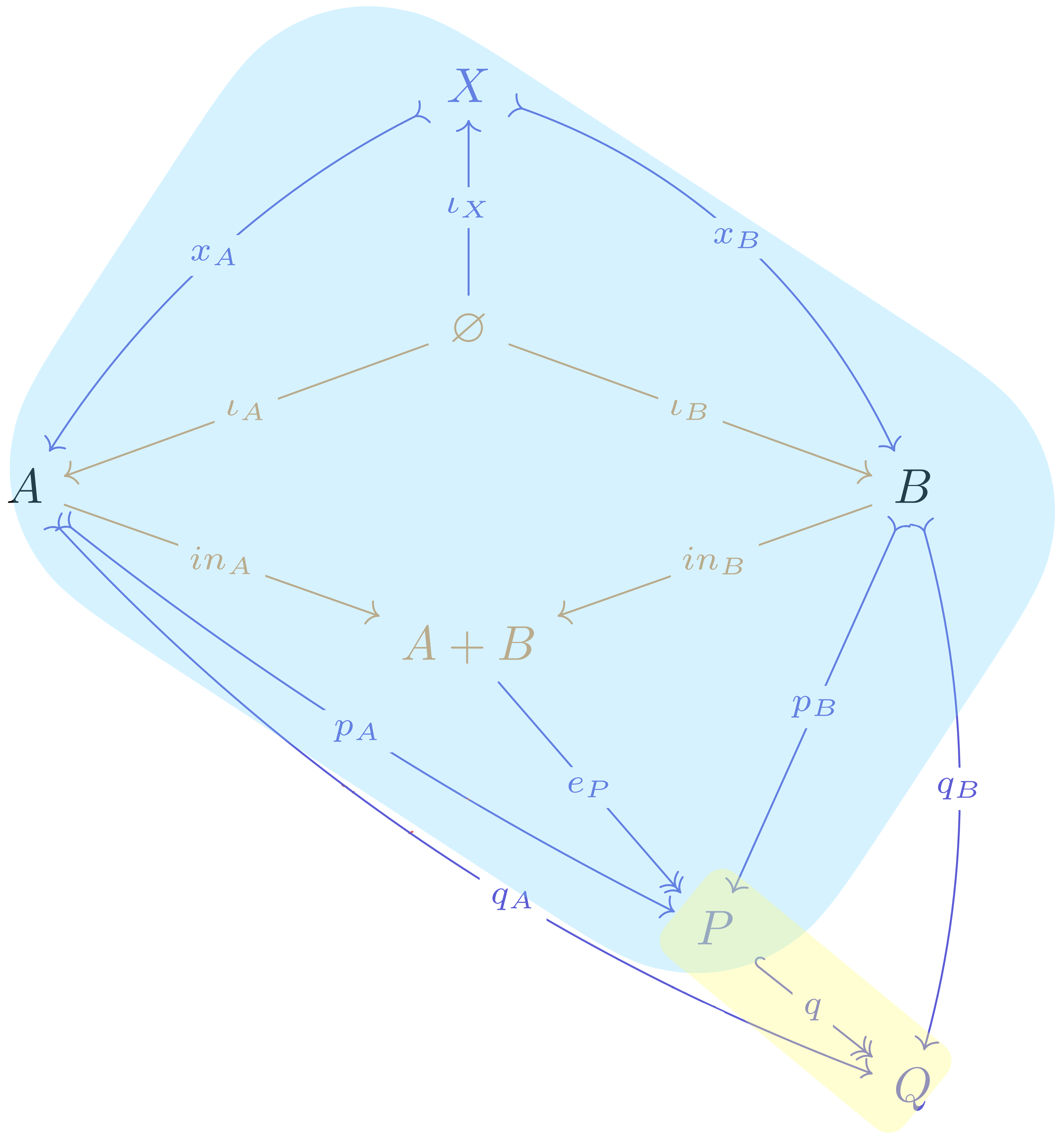


# Multi-sums

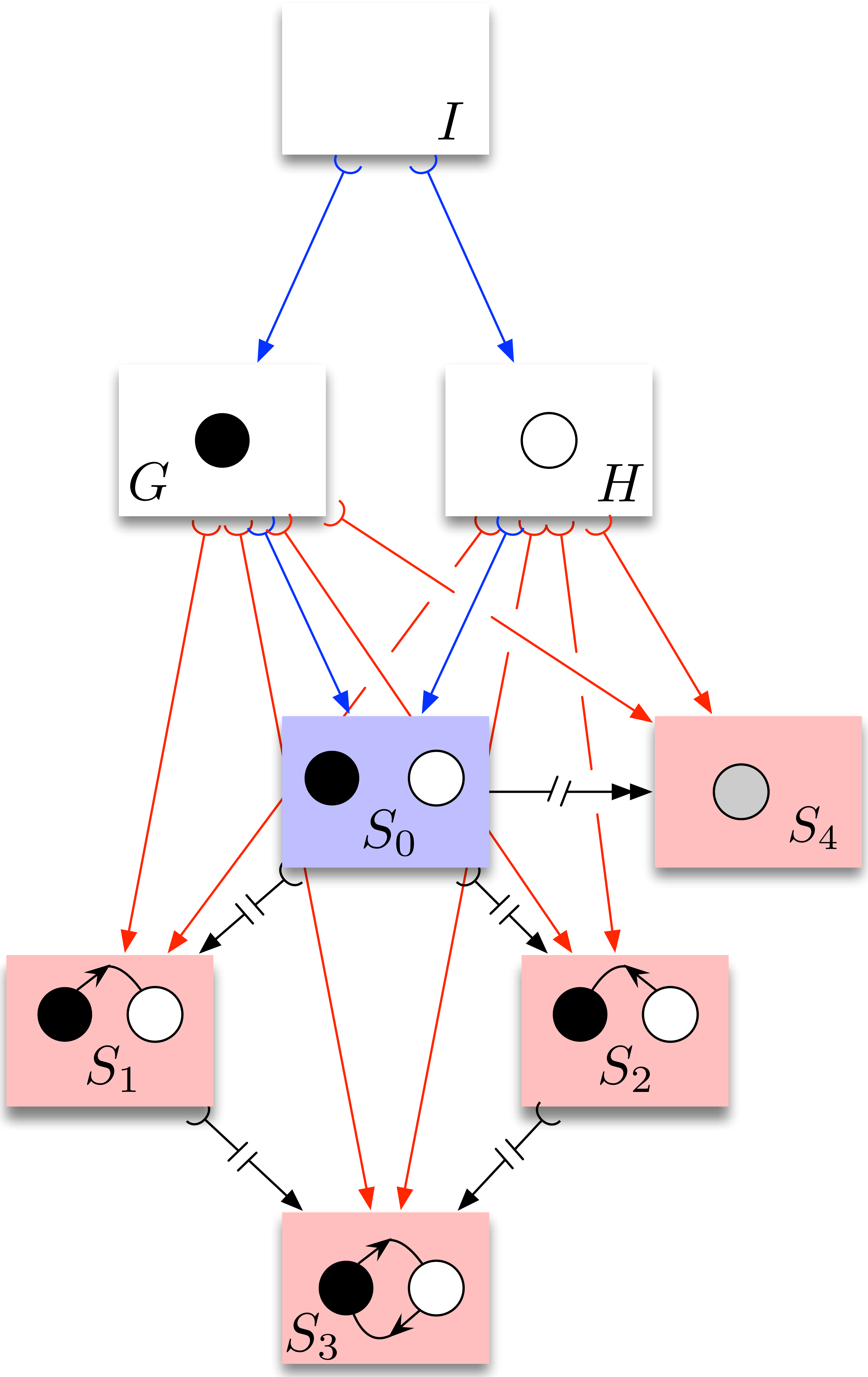
## Lemma

If $\mathbf{C}$ is a quasi-topos, the multi-sum $\sum_{\mathcal{M}}(A, B)$ arises from the epi-$\mathcal{M}$-factorization of $\mathbf{C}$ (for $\mathcal{M} = \mathsf{rm}(\mathbf{C})$).

- **Existence:** Let $A \xrightarrow{in_A} A + B \xleftarrow{in_B} B$ be the disjoint union of A and B. Then for any cospan $A \xrightarrow{a} Z \xleftarrow{b} B$ with $a, b \in \mathcal{M}$, the epi-$\mathcal{M}$-factorization of the induced arrow $A + B \xrightarrow{[a,b]} Z$ into an epimorphism $A + B \xrightarrow{e} Y$ and an $\mathcal{M}$-morphism $Y \xrightarrow{m} Z$ yields a cospan $(y_A = e \circ in_A, y_B = e \circ in_B)$, which by the decomposition property of $\mathcal{M}$-morphisms is a cospan of $\mathcal{M}$-morphisms.
- **Construction:** For objects $A, B \in obj(\mathbf{C})$, every element $A \xrightarrow{q_A} Q \xleftarrow{q_B} B$ in $\sum_{\mathcal{M}}(A, B)$ is obtained from a pushout of some span $A \xleftarrow{x_A} X \xrightarrow{x_B} B$ with $x_A, x_B \in \mathcal{M}$ and a morphism $P \xrightarrow{q} Q$ in $mono(\mathbf{C}) \cap epi(\mathbf{C})$.



# Multi-sums in SGraph



# Plan of the talk

1. **Quasi-topoi** in rewriting theory
2. **Prerequisites** for non-linear rewriting
3. **Non-linear DPO rewriting**
4. ***Non-linear SqPO rewriting***
5. Conclusion and outlook

# Concurrent rule composition for non-linear DPO rewriting

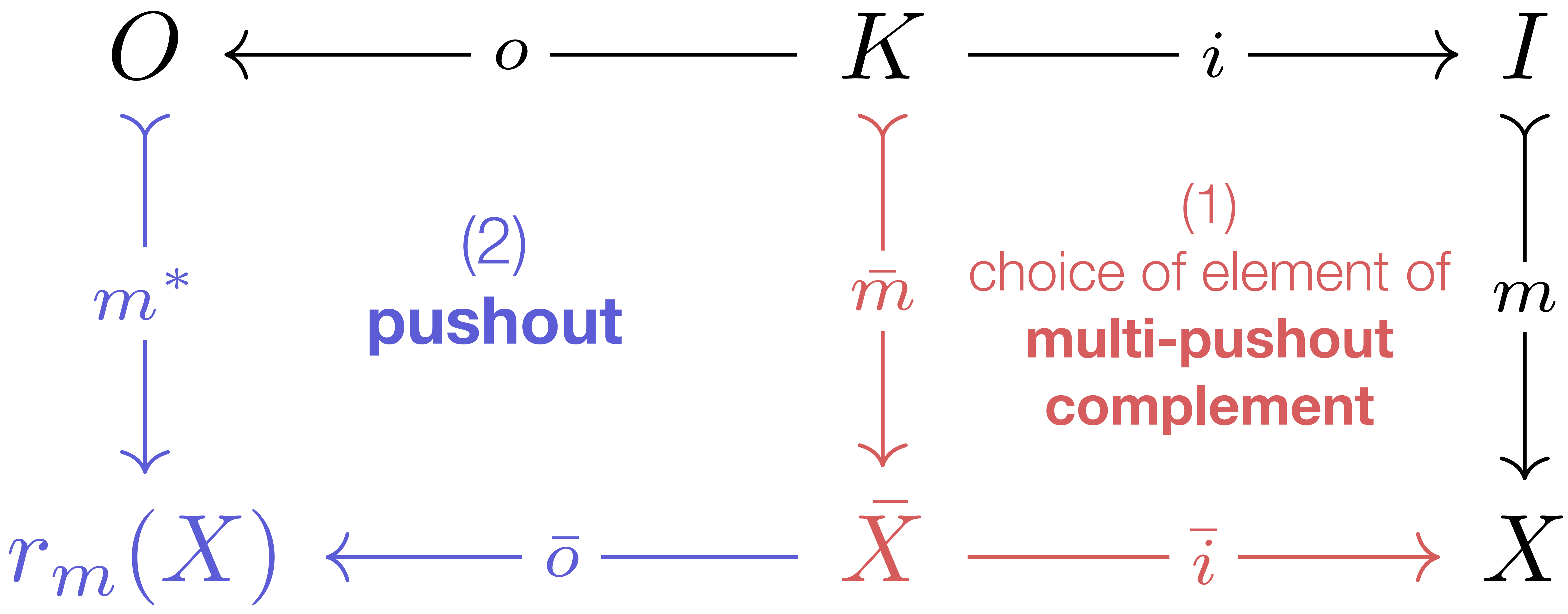
## Definition

General DPO-rewriting semantics over an rm-adhesive category **C**:

- The set of DPO-admissible matches of a rule rule $r = (O \leftarrow K \rightarrow I) \in \mathsf{span}(\mathbf{C})$ into an object $X \in \mathsf{obj}(\mathbf{C})$ is defined as

$$M_r^{DPO}(X) := \{(m, \bar{m}, \bar{i}) \mid m \in \mathsf{rm}(\mathbf{C}) \wedge (\bar{m}, \bar{i}) \in \mathcal{P}(i, m)\}\,.$$

  A DPO-type direct derivation of $X \in \mathsf{obj}(\mathbf{C})$ with rule $r$ along $m \in M_r^{DPO}(X)$ is defined as a diagram in (i), where (1) is the multi-POC element chosen as part of the data of the match, while (2) is formed as a pushout.



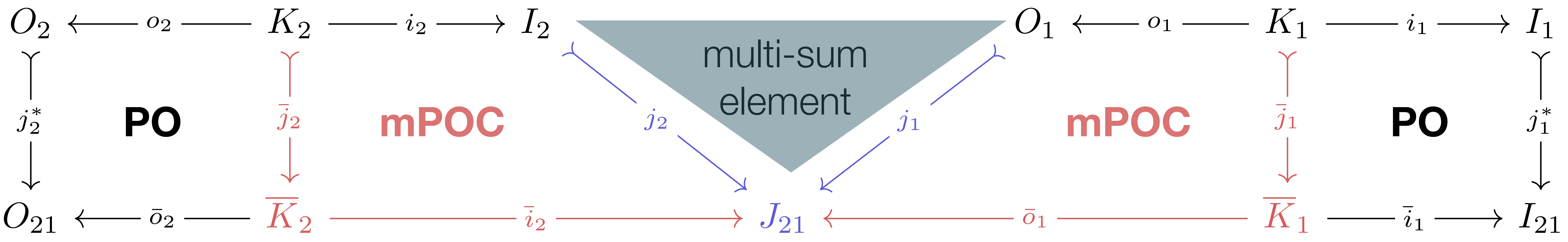

(i)

# Concurrent rule composition for non-linear DPO rewriting

- The set of DPO-type admissible matches of rules $\mathsf{r}_2, \mathsf{r}_1 \in \mathsf{span}(\mathbf{C})$ (also referred to as *dependency relations*) is defined as

$$\mathcal{M}^{DPO}_{\mathsf{r}_2}(\mathsf{r}_1) := \{(\mathsf{j}_2, \mathsf{j}_1, \bar{\mathsf{j}}_2, \bar{\mathsf{i}}_2, \bar{\mathsf{j}}_1, \bar{\mathsf{o}}_1) \mid (\mathsf{j}_2, \mathsf{j}_1) \in \textstyle\sum_{\mathcal{M}}(\mathsf{I}_2, \mathsf{O}_1) \wedge (\bar{\mathsf{j}}_2, \bar{\mathsf{i}}_2) \in \mathcal{P}(\mathsf{i}_2, \mathsf{j}_2) \wedge (\bar{\mathsf{j}}_1, \bar{\mathsf{o}}_1) \in \mathcal{P}(\mathsf{o}_1, \mathsf{j}_1)\}\big/_{\sim},$$

where equivalence is defined up to the compatible universal isomorphisms of multi-sums and multi-POCs.



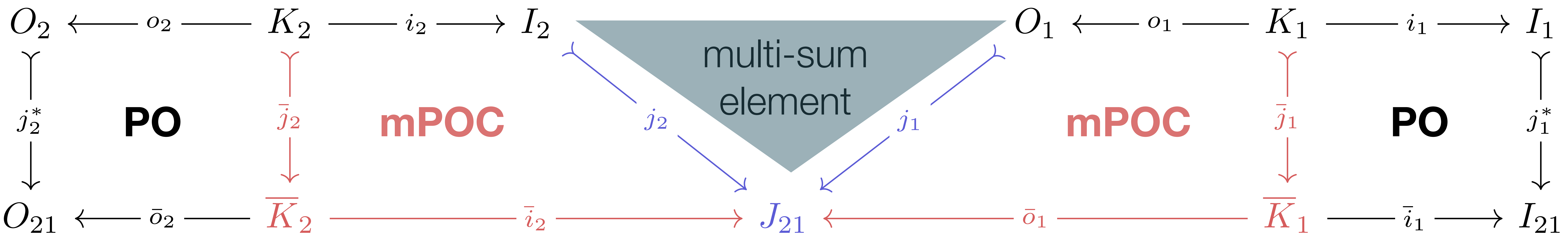

(ii)

# Concurrent rule composition for non-linear DPO rewriting

- The set of DPO-type admissible matches of rules $r_2, r_1 \in \mathsf{span}(\mathbf{C})$ (also referred to as *dependency relations*) is defined as

$$\mathcal{M}^{DPO}_{r_2}(r_1) := \{(j_2, j_1, \bar{j}_2, \bar{i}_2, \bar{j}_1, \bar{o}_1) \mid (j_2, j_1) \in \textstyle\sum_{\mathcal{M}}(I_2, O_1) \wedge (\bar{j}_2, \bar{i}_2) \in \mathcal{P}(i_2, j_2) \wedge (\bar{j}_1, \bar{o}_1) \in \mathcal{P}(o_1, j_1)\}/\sim\,,$$

where equivalence is defined up to the compatible universal isomorphisms of multi-sums and multi-POCs.




(ii)

- A DPO-type rule composition of two general rules $r_1, r_2 \in \mathsf{span}(\mathbf{C})$ along an admissible match $\mu \in \mathcal{M}^{DPO}_{r_2}(r_1)$ is defined via a diagram as in (ii), where $(1_2)$ and $(1_1)$ are the multi-POC elements chosen as part of the data of the match, while $(2_2)$ and $(2_1)$ are pushouts. We then define the composite rule via span composition:

$$r_2 \overset{\mu}{\blacktriangleleft} r_1 := (O_{21} \leftarrow \overline{K}_2 \rightarrow J_{21}) \circ (J_{21} \leftarrow \overline{K}_1 \rightarrow I_{21})$$

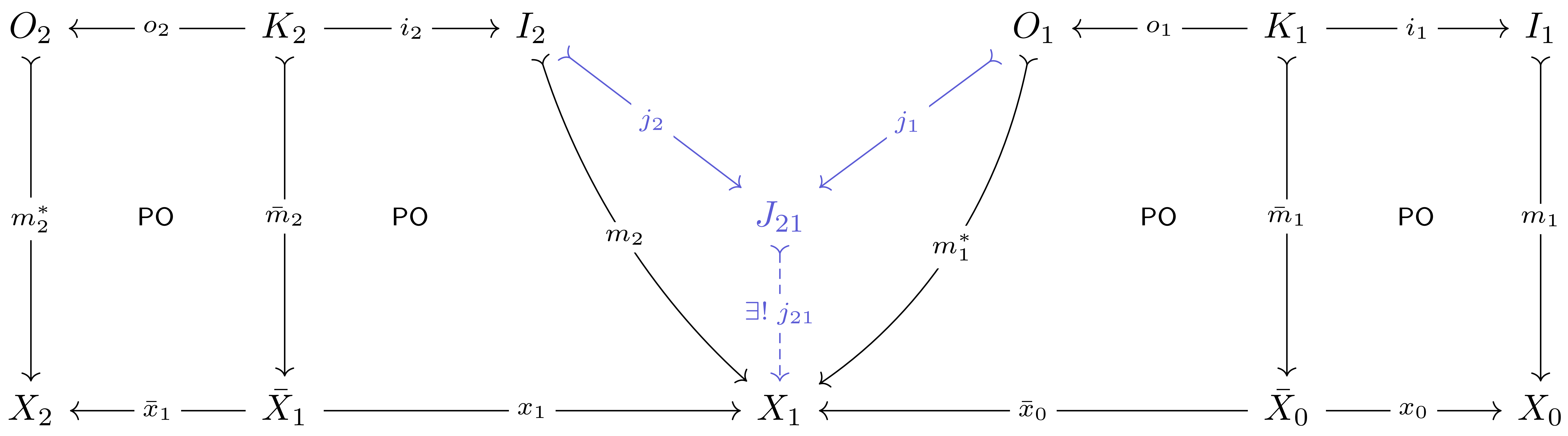
# Concurreny theorem for non-linear DPO rewriting

## Theorem

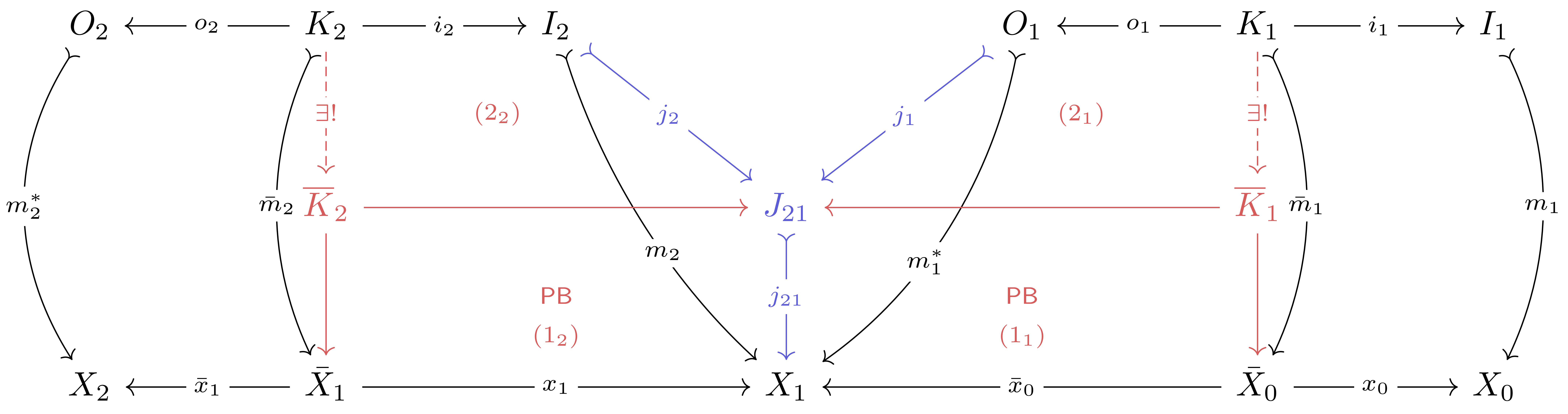
Let $\mathbf{C}$ be an rm-adhesive category, let $X_0 \in \mathtt{obj}(\mathbf{C})$ be an object, and let $r_2, r_1 \in \mathtt{span}(\mathbf{C})$ be (generic) spans in $\mathbf{C}$.

- Synthesis: For every pair of admissible matches $m_1 \in M^{DPO}_{r_1}(X_0)$ and $m_2 \in M^{DPO}_{r_2}(r_{1_{m_1}}(X_0))$, there exist an admissible match $\mu \in \mathcal{M}^{DPO}_{r_2}(r_1)$ and an admissible match $m_{21} \in M^{DPO}_{r_{21}}(X_0)$ (for $r_{21}$ the composite of $r_2$ with $r_1$ along $\mu$) such that $r_{21_{m_{21}}}(X_0) \cong r_{2_{m_2}}(r_{1_{m_1}}(X_0))$.
- Analysis: For every pair of admissible matches $\mu \in \mathcal{M}^{DPO}_{r_2}(r_1)$ and $m_{21} \in M^{DPO}_{r_{21}}(X_0)$ (for $r_{21}$ the composite of $r_2$ with $r_1$ along $\mu$), there exists a pair of admissible matches $m_1 \in M^{DPO}_{r_1}(X_0)$ and $m_2 \in M^{SqPO}_{r_2}(r_{1_{m_1}}(X_0))$ such that $r_{2_{m_2}}(r_{1_{m_1}}(X_0)) \cong r_{21_{m_{21}}}(X_0)$.
- Compatibility: If in addition $\mathbf{C}$ is finitary, the sets of pairs of matches $(m_1, m_2)$ and $(\mu, m_{21})$ are isomorphic if they are suitably quotiented by universal isomorphisms, i.e., by universal isomorphisms of $X_1 = r_{1_{m_1}}(X_0)$ and $X_2 = r_{2_{m_2}}(X_1)$ for the set of pairs of matches $(m_1, m_2)$, and by the universal isomorphisms of multi-sums and multi-POCs for the set of pairs of matches $(\mu, m_{21})$, respectively.

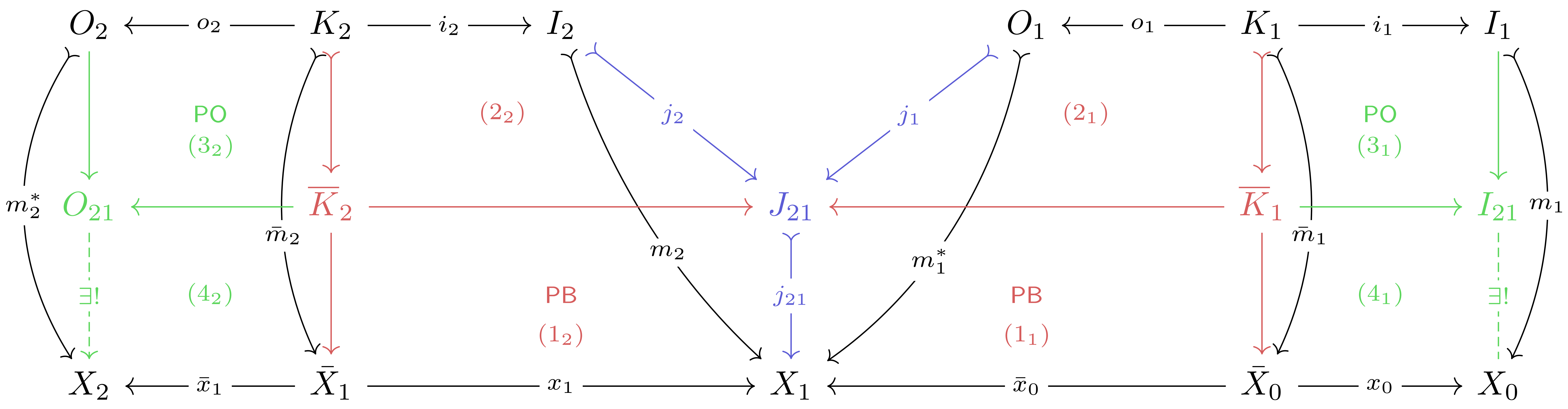
# Proof of the **synthesis part**



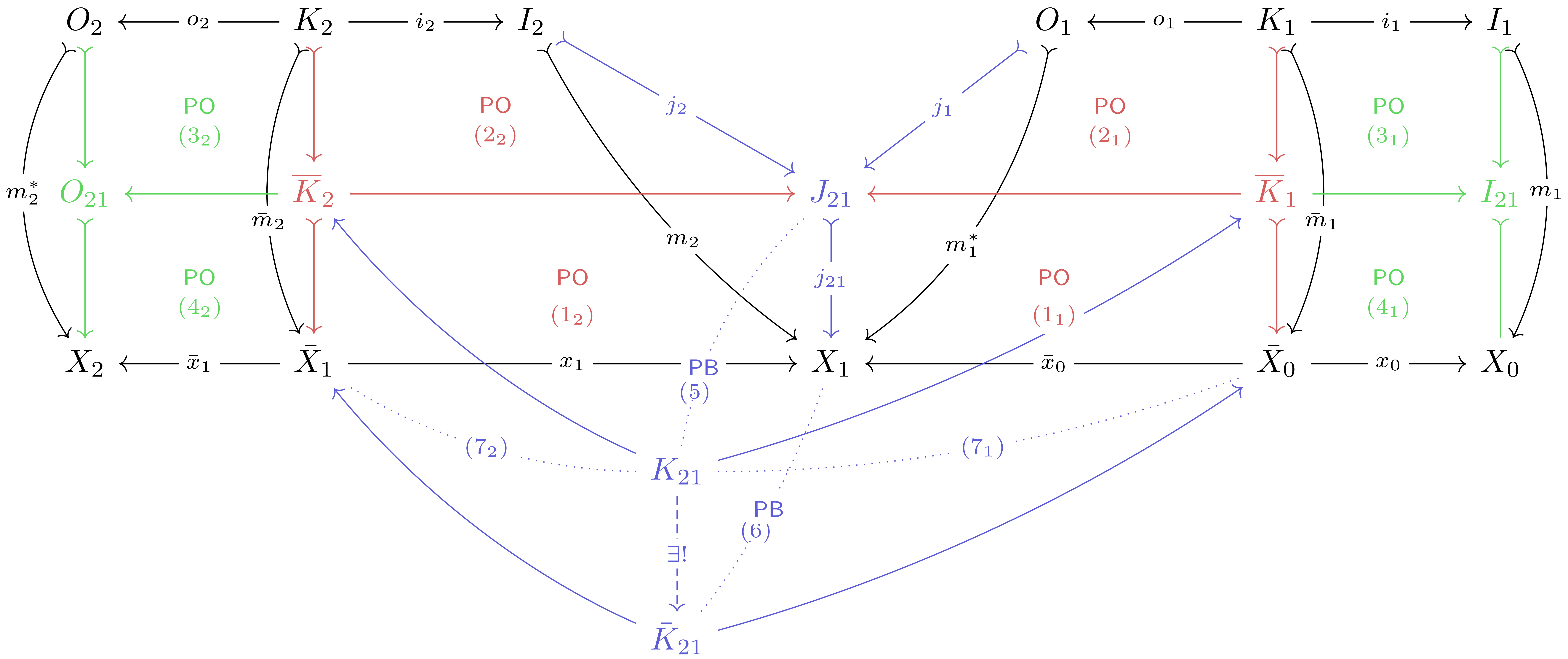
# Proof of the **synthesis part**



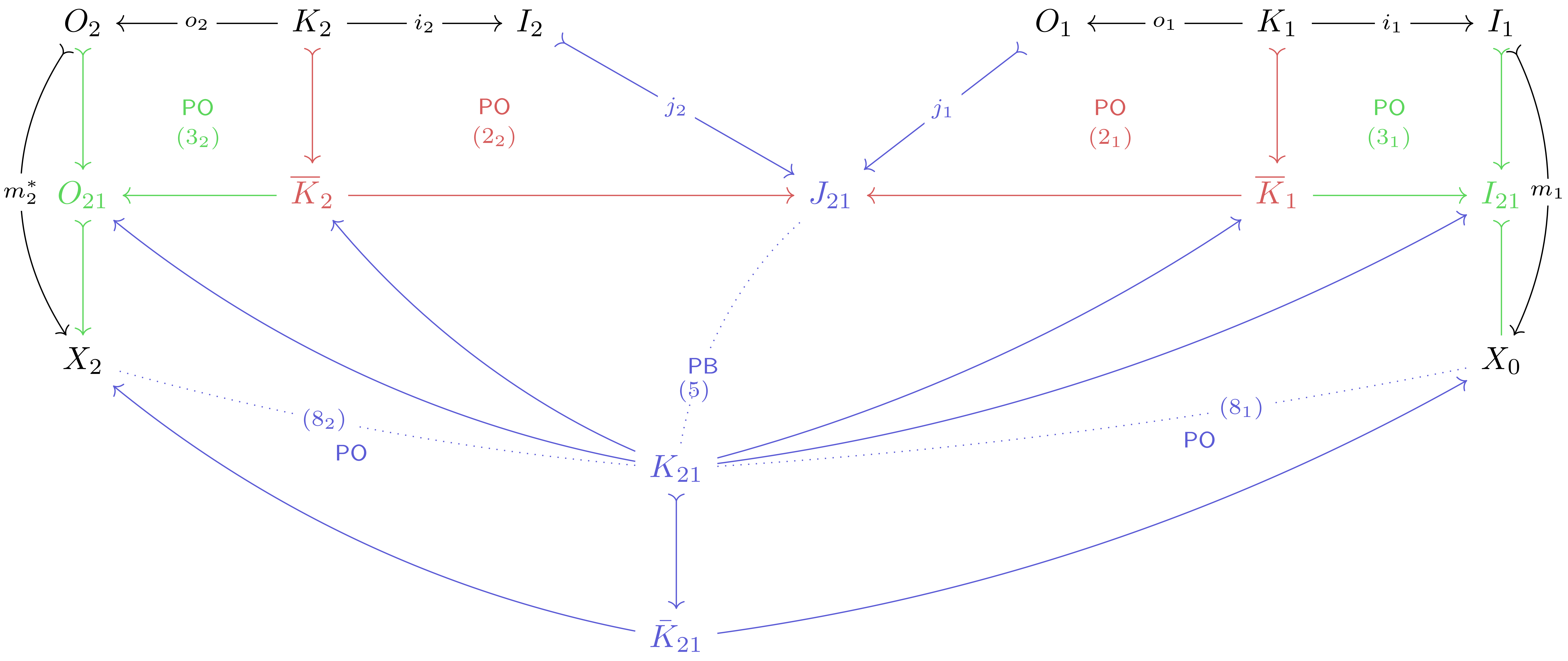
# Proof of the **synthesis part**



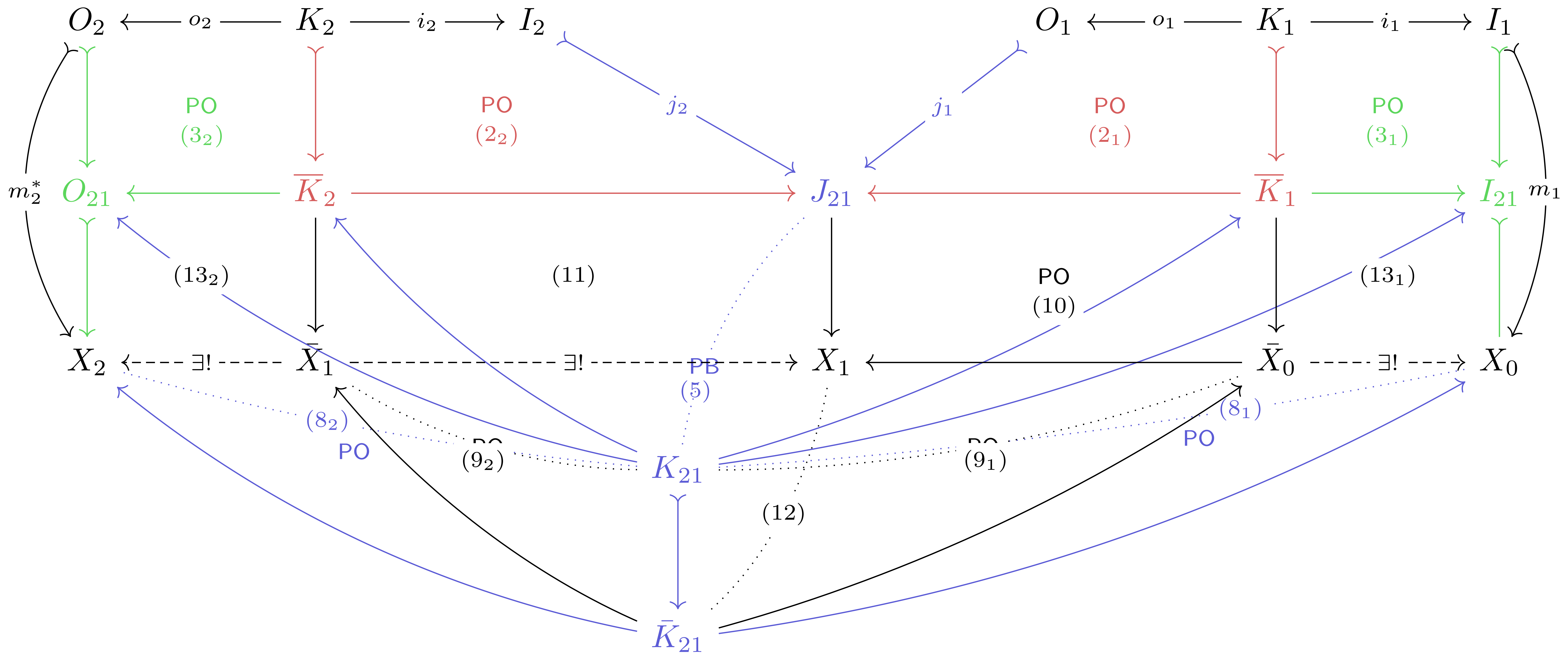
# Proof of the **synthesis part**



# Proof of the **analysis part**



# Proof of the **analysis part**



## Plan of the talk

1. **Quasi-topoi** in rewriting theory
2. **Prerequisites** for non-linear rewriting
3. **Non-linear DPO rewriting**
4. ***Non-linear SqPO rewriting***
5. Conclusion and outlook

# Concurrent rule composition for non-linear SqPO rewriting

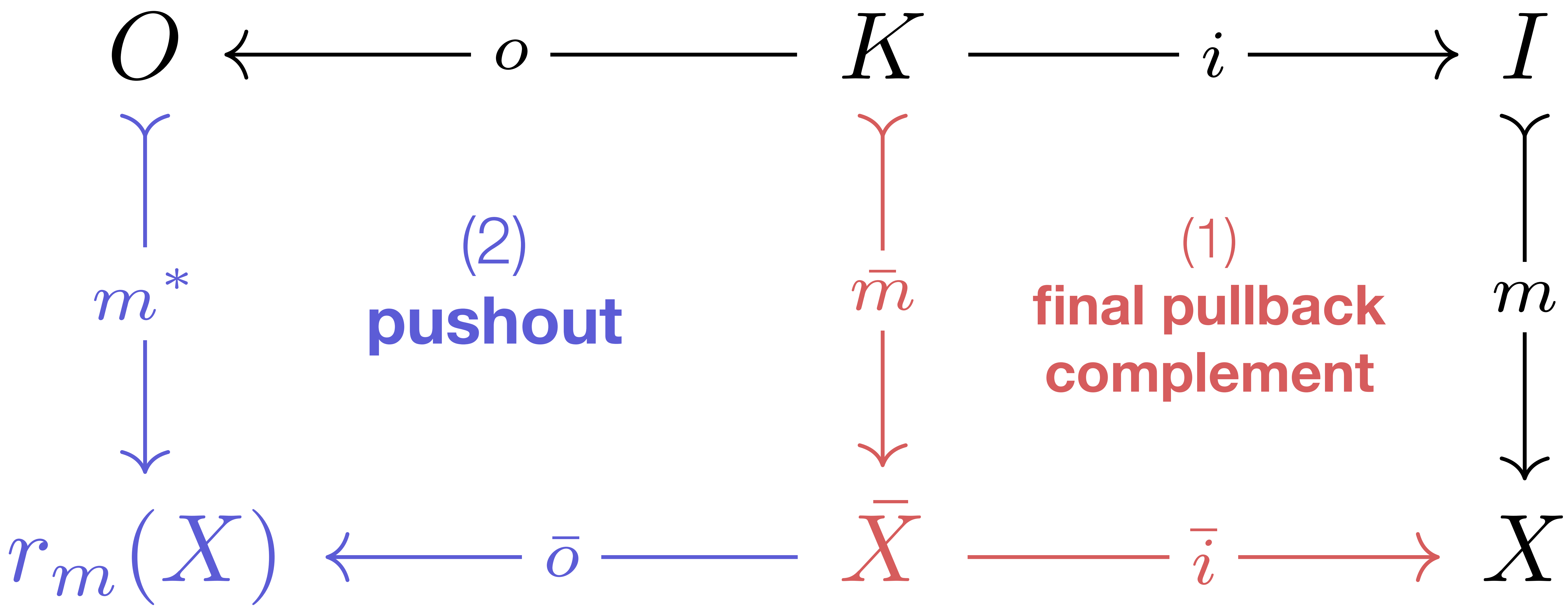
## Definition

General SqPO-rewriting semantics over a quasi-topos **C**:

- The set of SqPO-admissible matches of a rule rule $\mathsf{r} = (\mathsf{O} \leftarrow \mathsf{K} \rightarrow \mathsf{I}) \in \mathsf{span}(\mathbf{C})$ into an object $\mathsf{X} \in \mathsf{obj}(\mathbf{C})$ is defined as

$$\mathsf{M}_r^{SqPO}(X) := \{I \xrightarrow{m} X \mid m \in \mathsf{rm}(\mathbf{C})\}\,.$$

  A SqPO-type direct derivation of $\mathsf{X} \in \mathsf{obj}(\mathbf{C})$ with rule $\mathsf{r}$ along $\mathsf{m} \in \mathsf{M}_{\mathsf{r}}^{SqPO}(\mathsf{X})$ is defined as a diagram in (i), where (1) is formed as an FPC, while (2) is formed as a pushout.




(i)

# Concurrent rule composition for non-linear SqPO rewriting

- The set of SqPO-type admissible matches of rules $r_2, r_1 \in \mathsf{span}(\mathbf{C})$ (also referred to in the literature as dependency relations) is defined as

$$\mathcal{M}^{SqPO}_{r_2}(r_1) := \{(j_2, j_1, j_1, o_1, j_1, i_1, \iota_{21}) \mid (j_2, j_1) \in \sum\nolimits_{\mathcal{M}}(I_2, O_1) \wedge (j_1, o_1) \in \mathcal{P}(o_1, j_1) \wedge (j_1, i_1, \iota_{21}) \in \mathsf{FPA}(j_1, i_1)\}\diagup_{\sim},$$

where equivalence is defined up to the compatible universal isomorphisms of multi-sums, multi-POCs and FPAs (see below).

# Concurrent rule composition for non-linear SqPO rewriting

- The set of SqPO-type admissible matches of rules $\mathsf{r}_2, \mathsf{r}_1 \in \mathsf{span}(\mathbf{C})$ (also referred to in the literature as dependency relations) is defined as

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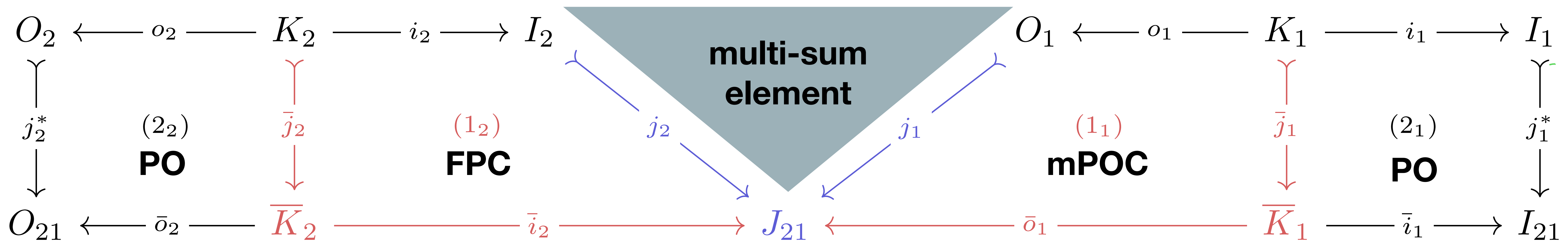
where equivalence is defined up to the compatible universal isomorphisms of multi-sums, multi-POCs and FPAs (see below).




(ii)

# Concurrent rule composition for non-linear SqPO rewriting

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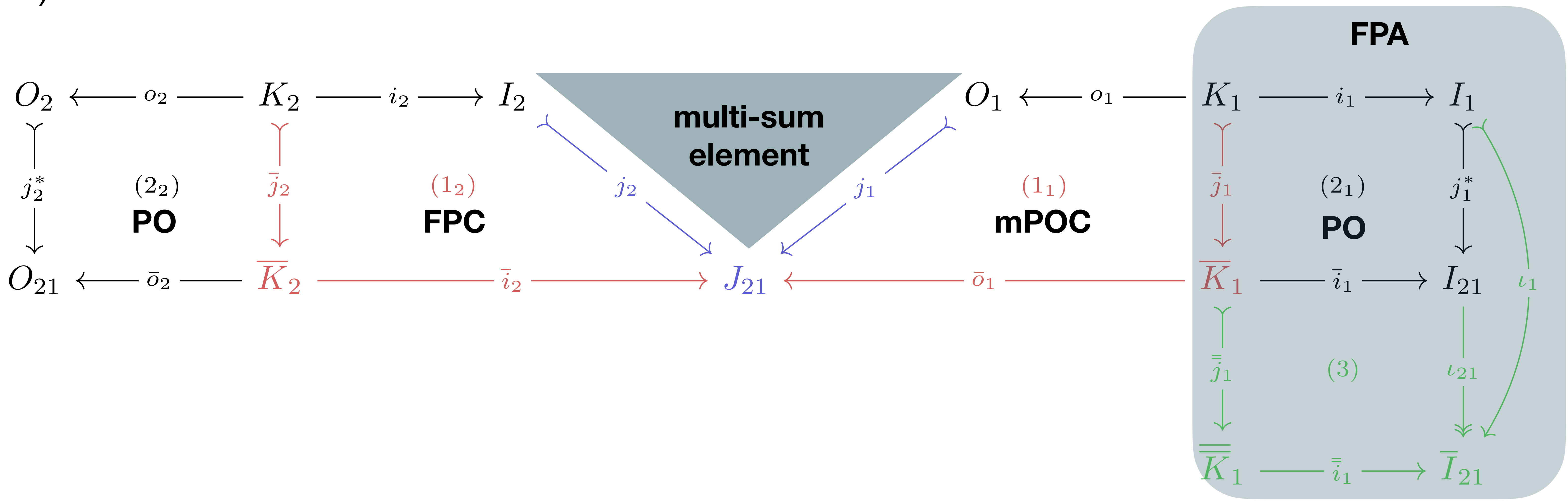
where equivalence is defined up to the compatible universal isomorphisms of multi-sums, multi-POCs and FPAs (see below).



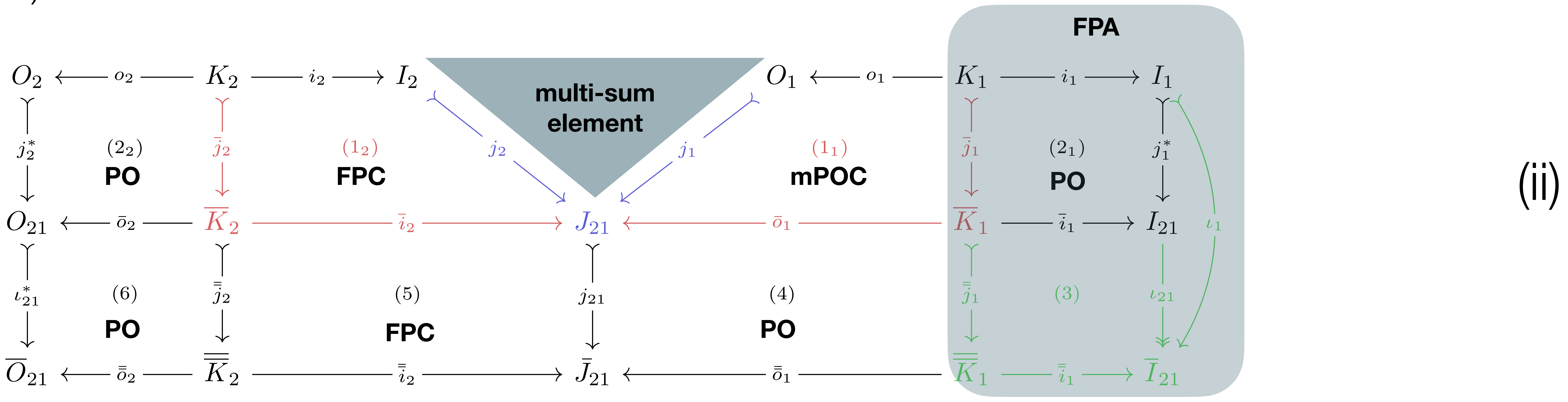

(ii)

# Concurrent rule composition for non-linear SqPO rewriting

- The set of SqPO-type admissible matches of rules $r_2, r_1 \in \mathsf{span}(\mathbf{C})$ (also referred to in the literature as dependency relations) is defined as

$$\mathcal{M}^{SqPO}_{r_2}(r_1) := \{(j_2, j_1, j_1, o_1, j_1, i_1, \iota_{21}) \mid (j_2, j_1) \in \sum\nolimits_{\mathcal{M}}(I_2, O_1) \wedge (j_1, o_1) \in \mathcal{P}(o_1, j_1) \wedge (j_1, i_1, \iota_{21}) \in \mathsf{FPA}(j_1, i_1)\}\big/_{\sim},$$

where equivalence is defined up to the compatible universal isomorphisms of multi-sums, multi-POCs and FPAs (see below).



# Concurrent rule composition for non-linear SqPO rewriting

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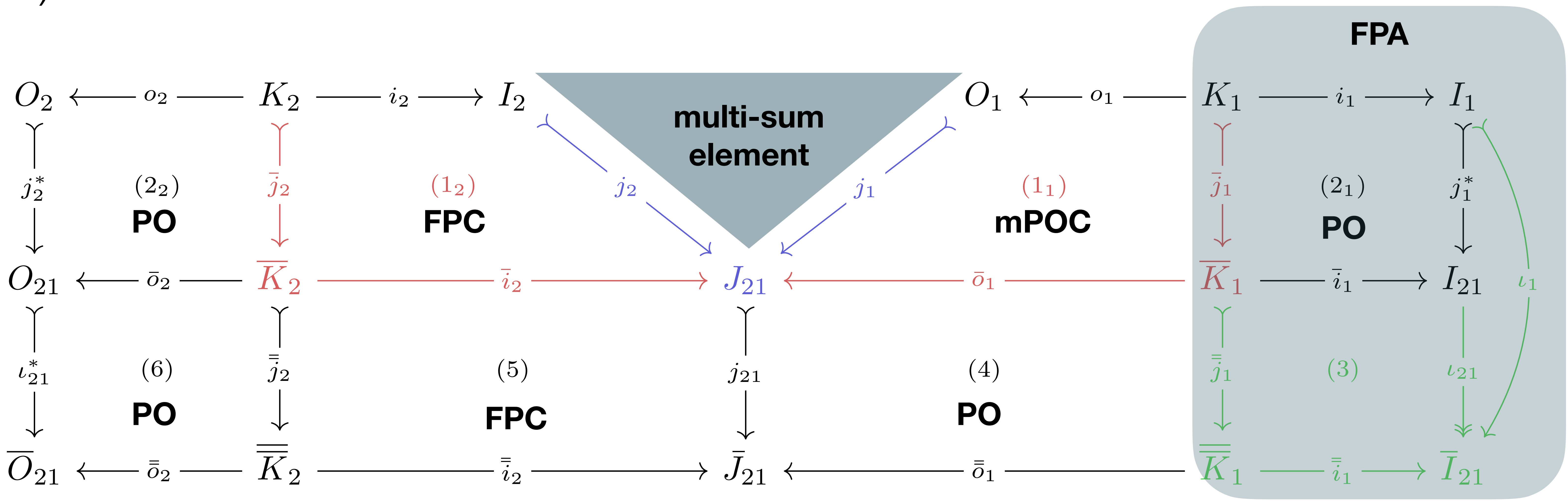
where equivalence is defined up to the compatible universal isomorphisms of multi-sums, multi-POCs and FPAs (see below).

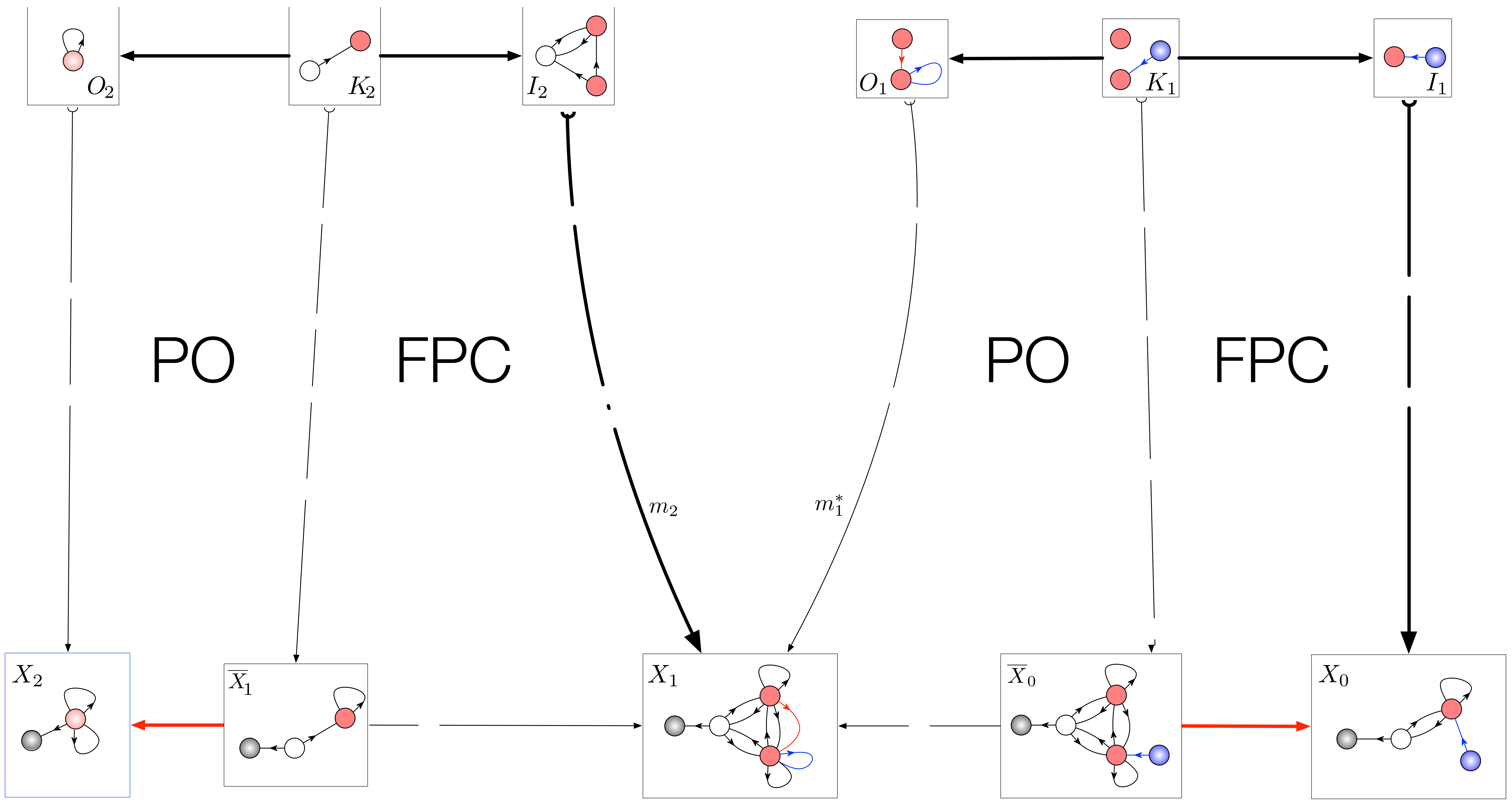



(ii)

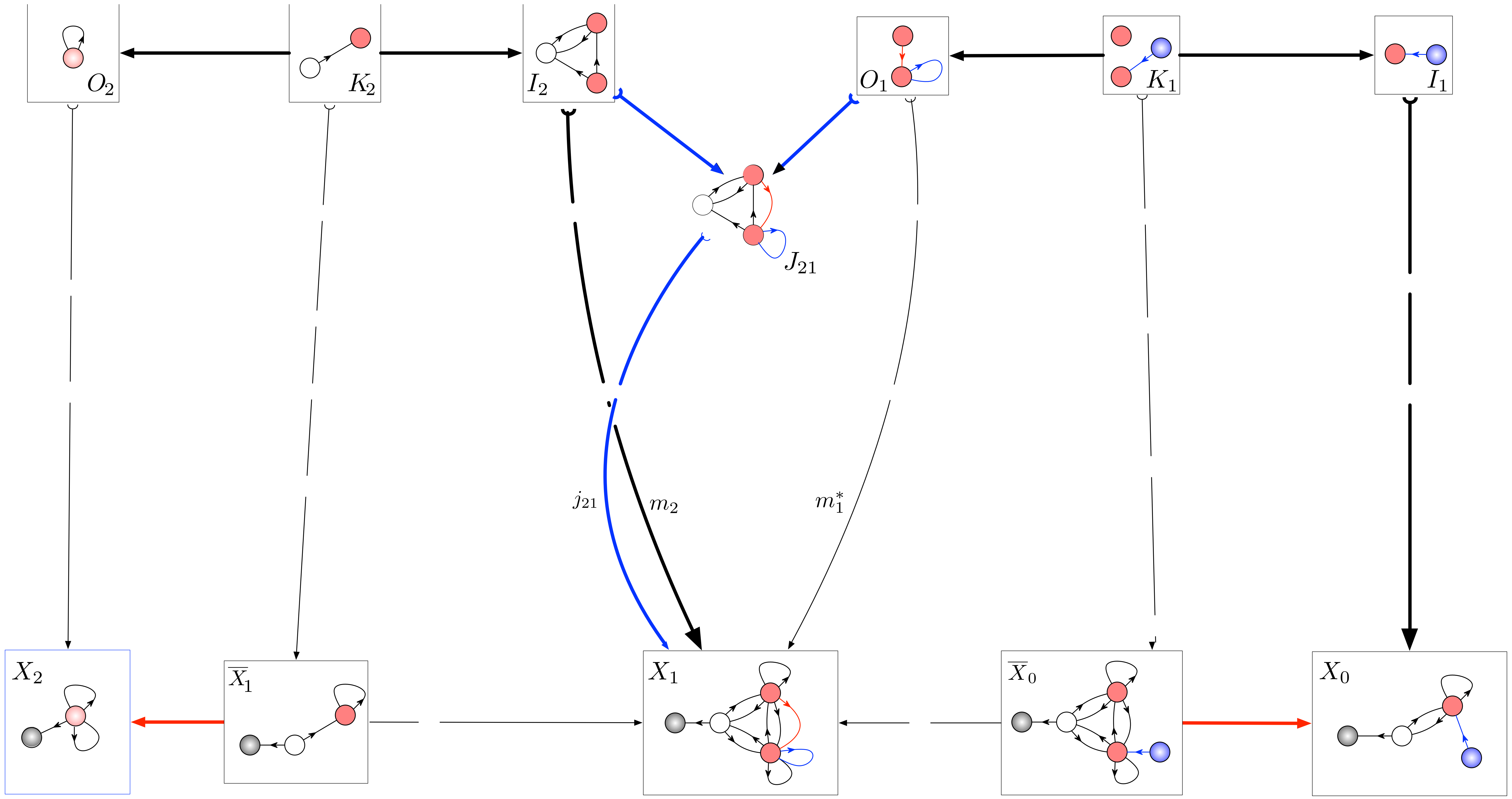
- An SqPO-type rule composition of two general rules $r_1, r_2 \in \mathsf{span}(\mathbf{C})$ along an admissible match $\mu \in \mathcal{M}^{SqPO}_{r_2}(r_1)$ is defined via a diagram as in (ii), where (going column-wise from the left) squares $(2_2)$, $(6)$, and $(4)$ are pushouts, $(1_1)$ is the multi-POC element specified as part of the data of the match, $(2_1)$ and $(3)$ form an FPA-diagram as per the data of the match, and finally $(1_2)$ and $(5)$ are FPCs. We then define the composite rule via span composition:

$$r_2 \overset{\mu}{\triangleleft} r_1 := (\overline{O}_{21} \leftarrow \overline{\overline{K}}_2 \rightarrow \overline{J}_{21}) \circ (\overline{J}_{21} \leftarrow \overline{\overline{K}}_1 \rightarrow \overline{I}_{21})$$

# **Concurrent rule composition** for **non-linear SqPO rewriting**

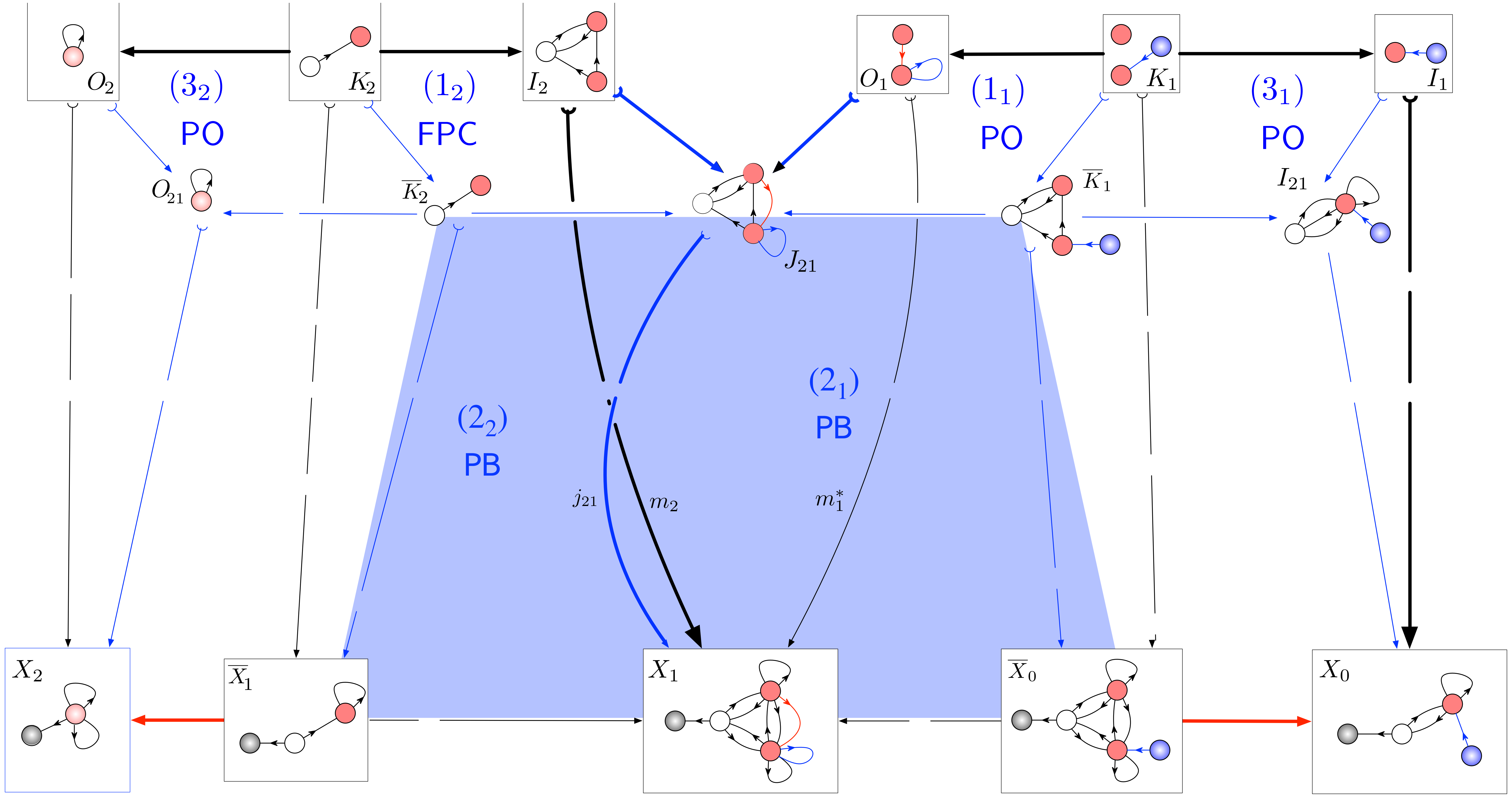


# **Concurrent rule composition** for **non-linear SqPO rewriting**



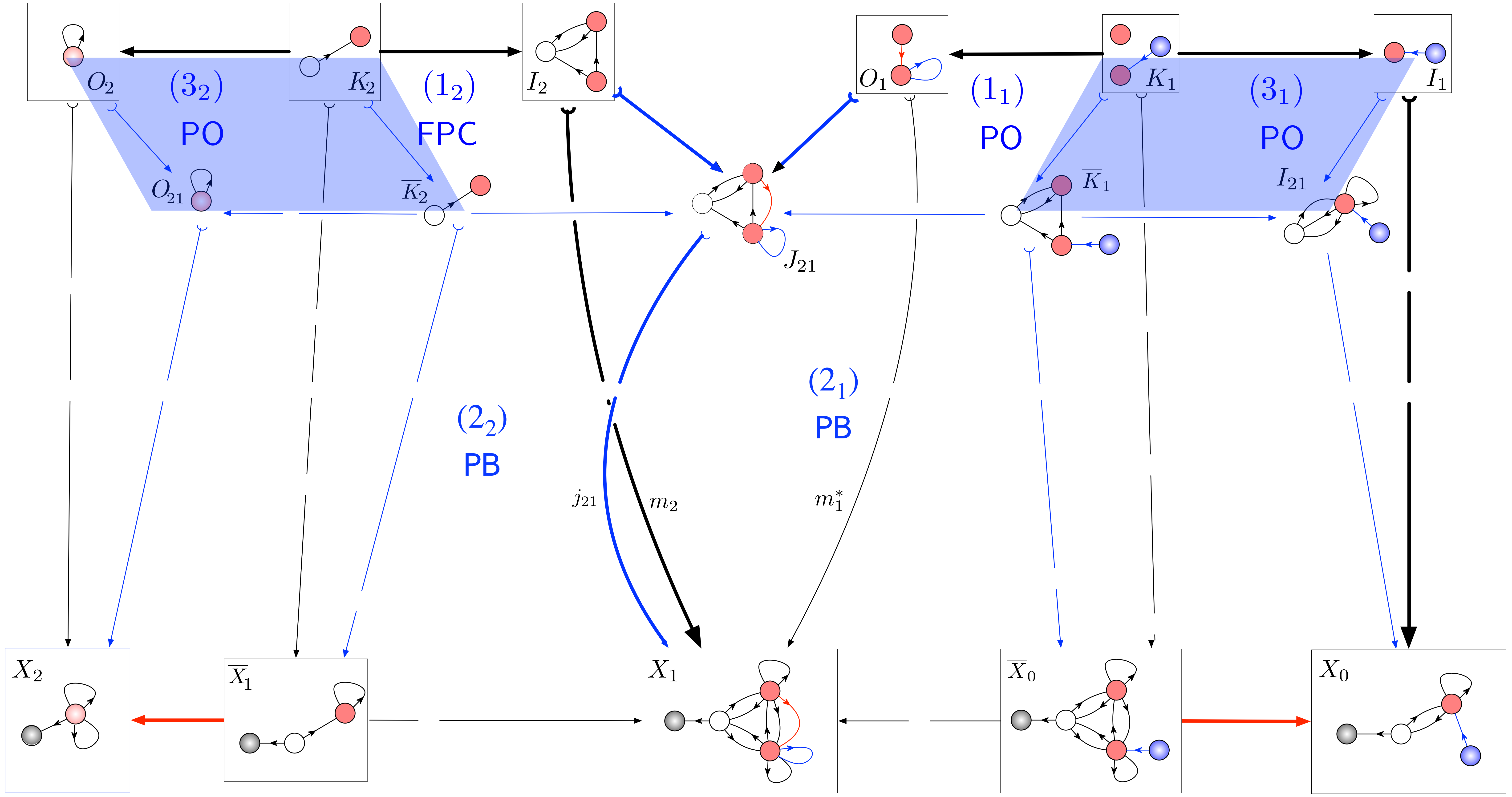

determine the **multi-sum element** $J_{21}$ (uniquely up to universal isomorphisms)

# **Concurrent rule composition** for **non-linear SqPO rewriting**




Take **pullbacks** to obtain squares $(2_2)$ and $(2_1)$ …

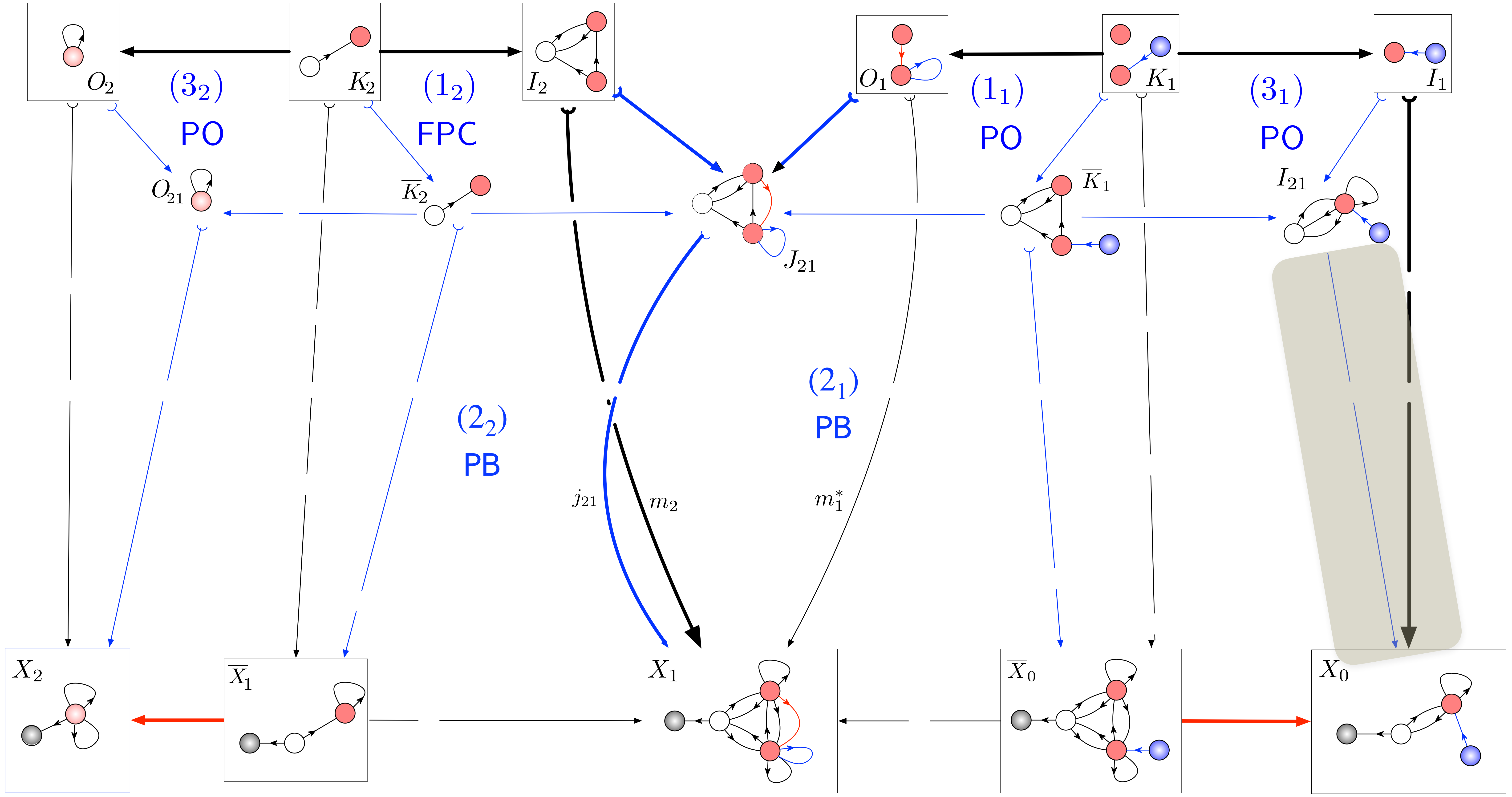
# **Concurrent rule composition** for **non-linear SqPO rewriting**




Take **pullbacks** to obtain squares $(2_2)$ and $(2_1)$ …

… then **pushouts** to obtain squares $(3_2)$ and $(3_1)$ …

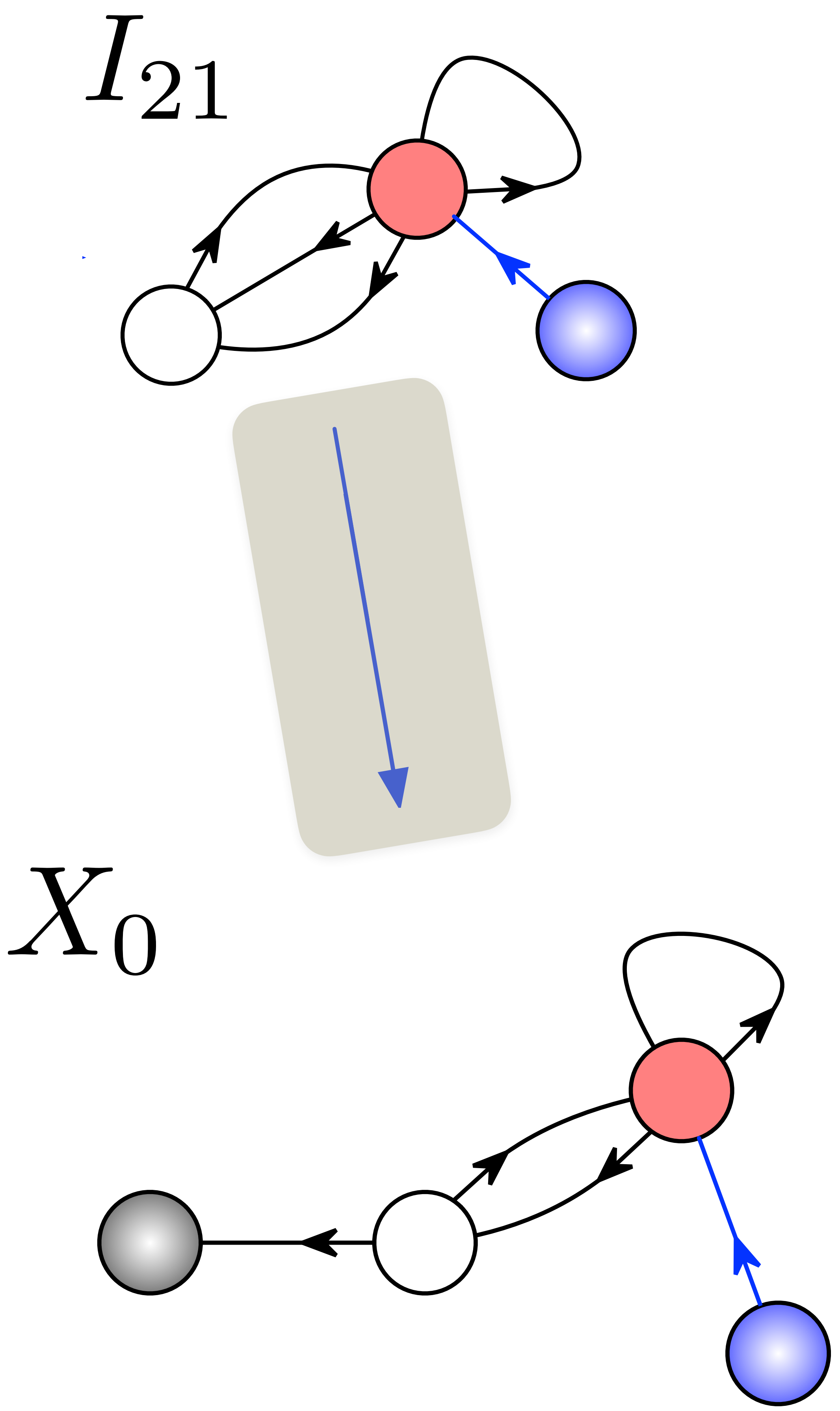
# Concurrent rule composition for non-linear SqPO rewriting

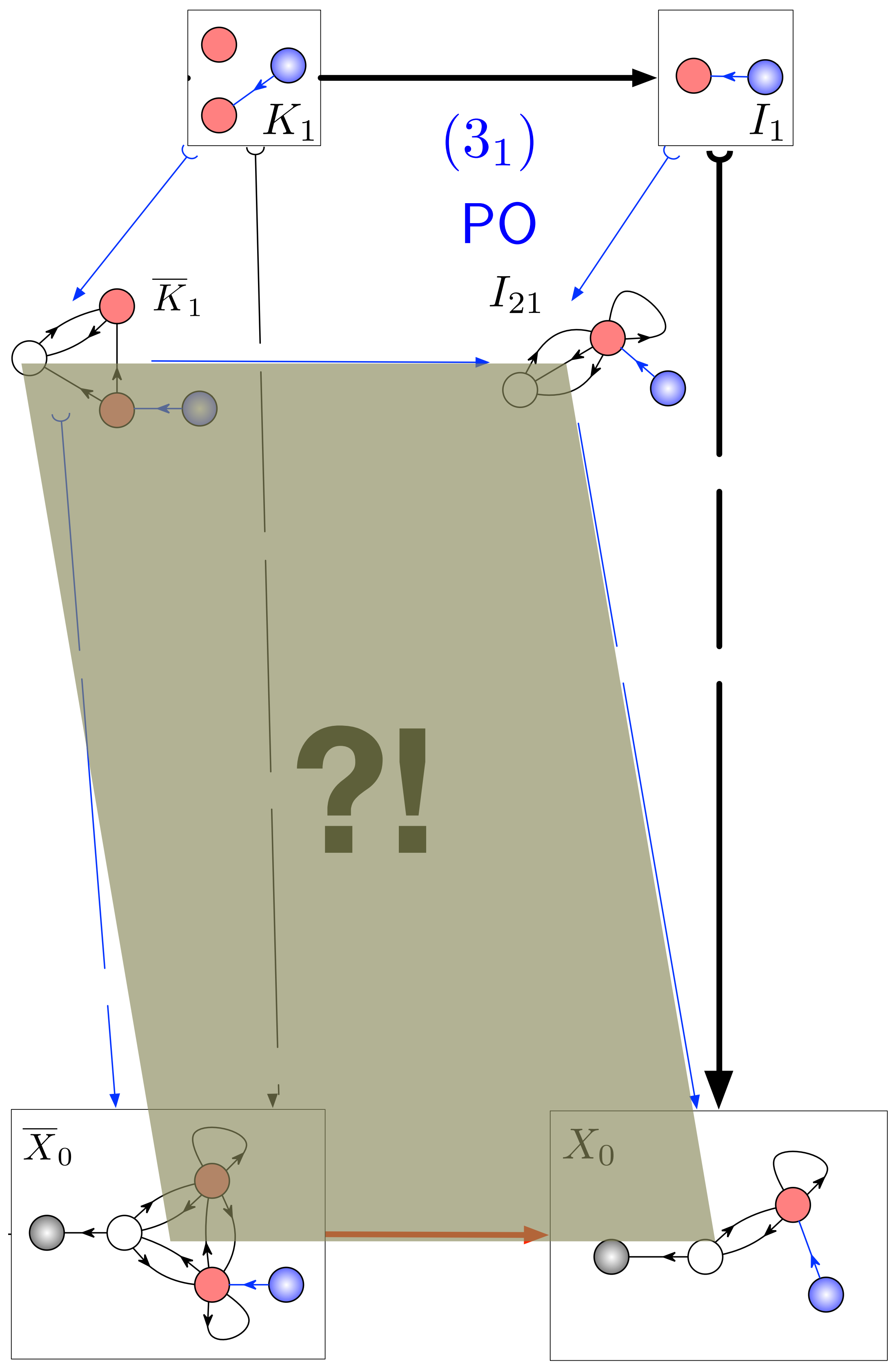



Take **pullbacks** to obtain squares $(2_2)$ and $(2_1)$ …

… then **pushouts** to obtain squares $(3_2)$ and $(3_1)$ …

# **Concurrent rule composition** for **non-linear SqPO rewriting**

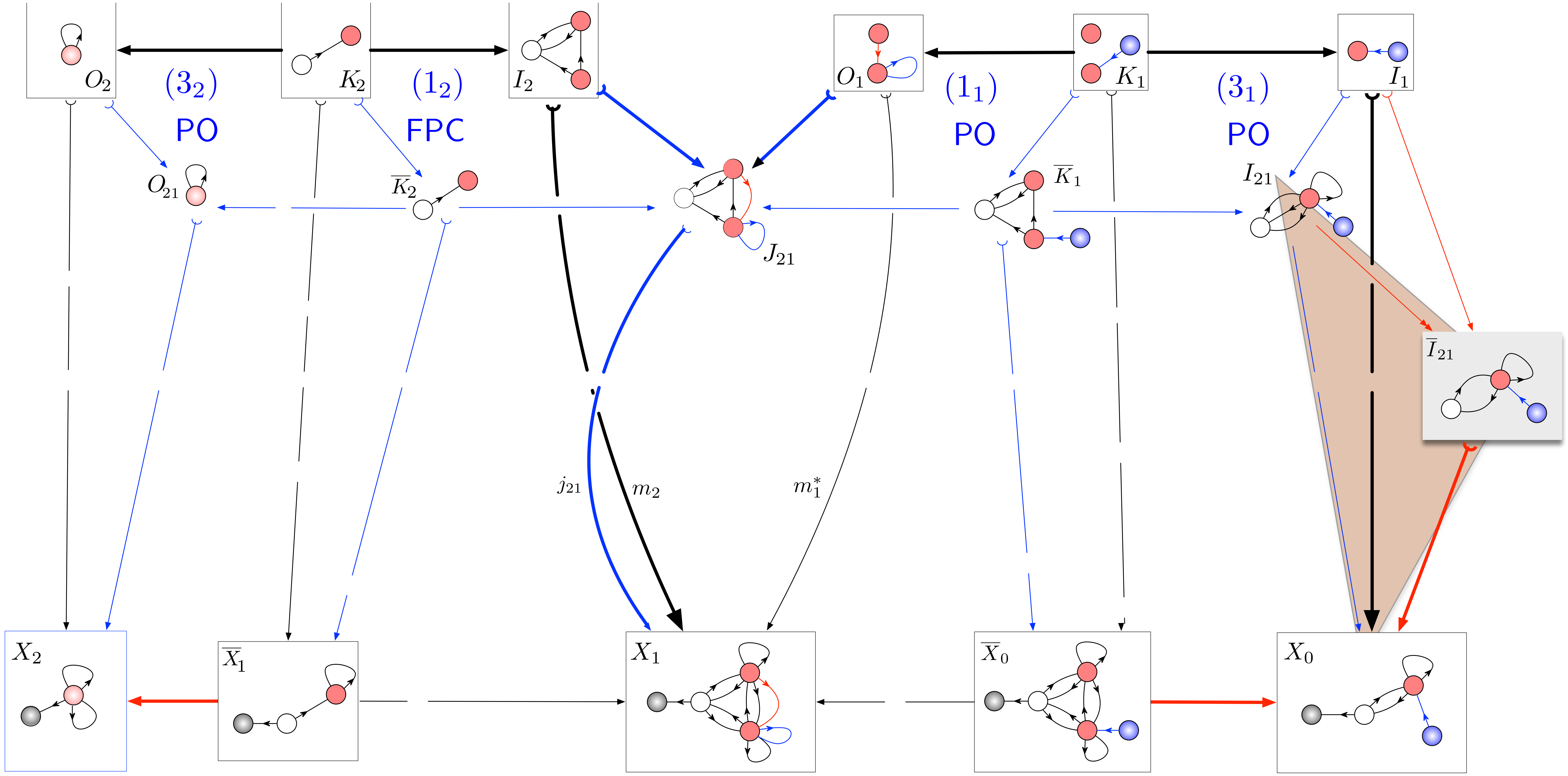



… **BUT** $I_{21} \to X_0$ is **not** a monomorphism …

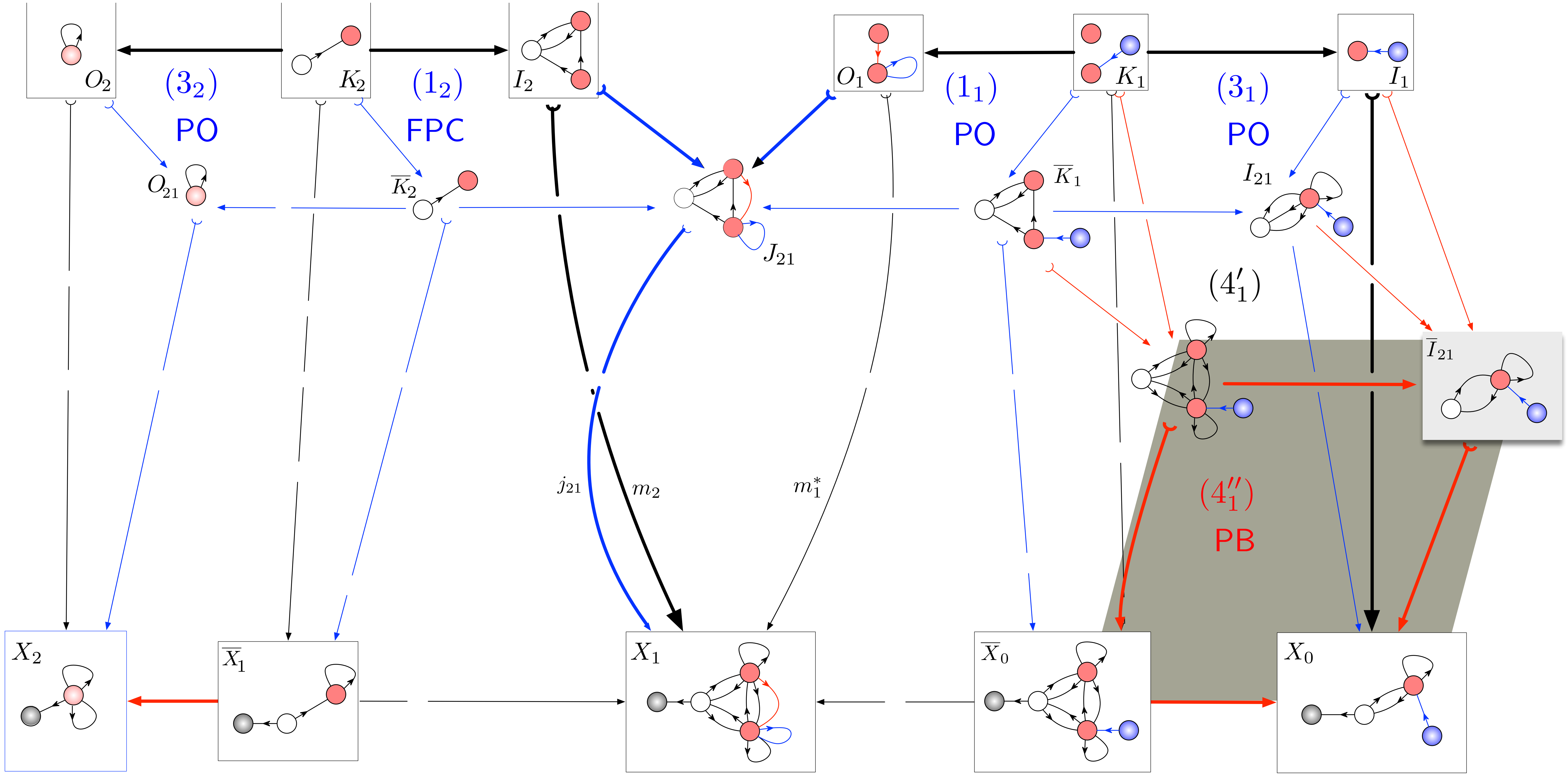
…and the square marked **?!** Is neither a pushout, FPC nor a pullback!

# **Concurrent rule composition** for **non-linear SqPO rewriting**



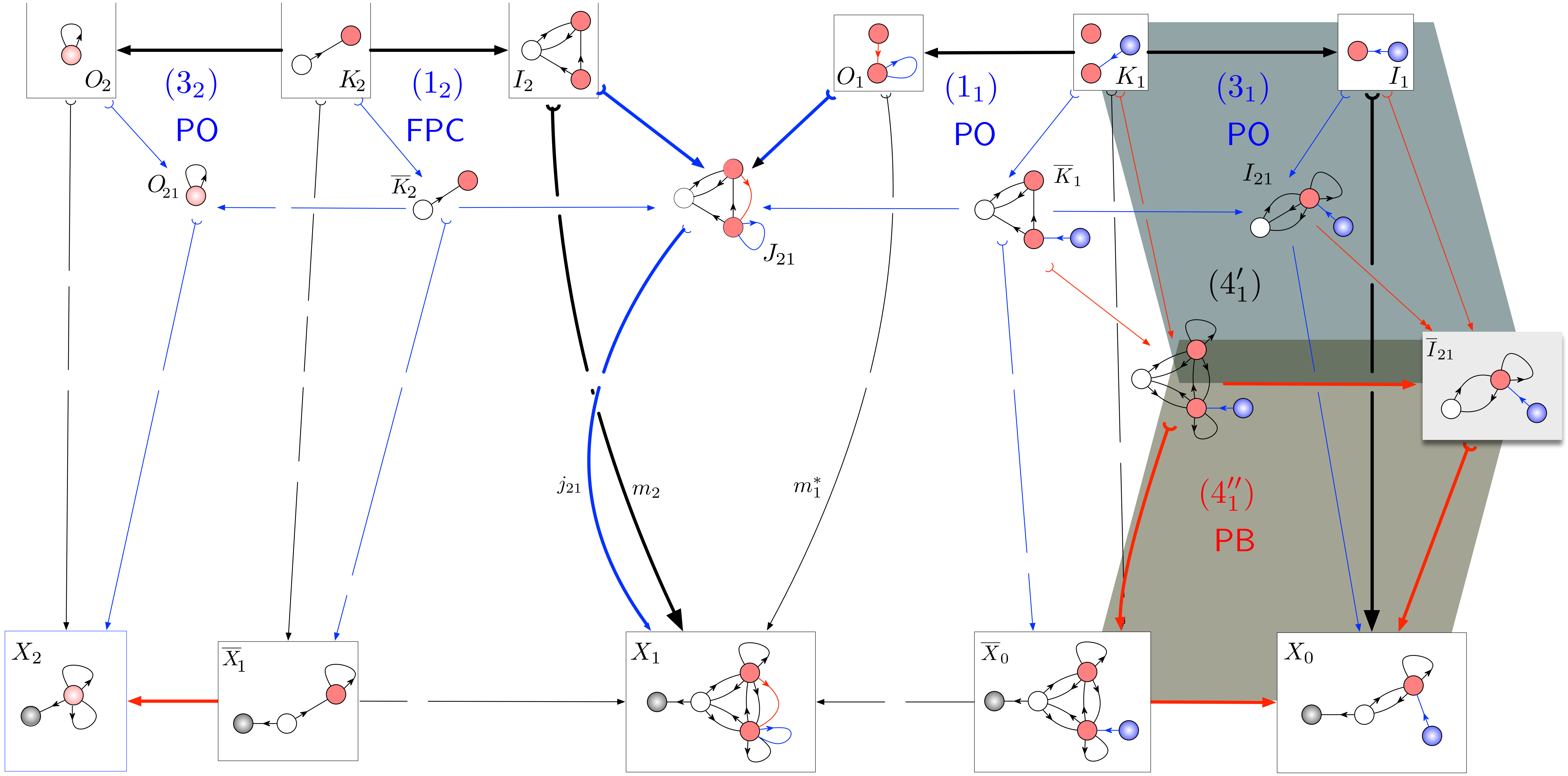

**Resolution:** form the **epi-regular mono-factorization** of $I_{21} \to X_0$

# Concurrent rule composition for non-linear SqPO rewriting



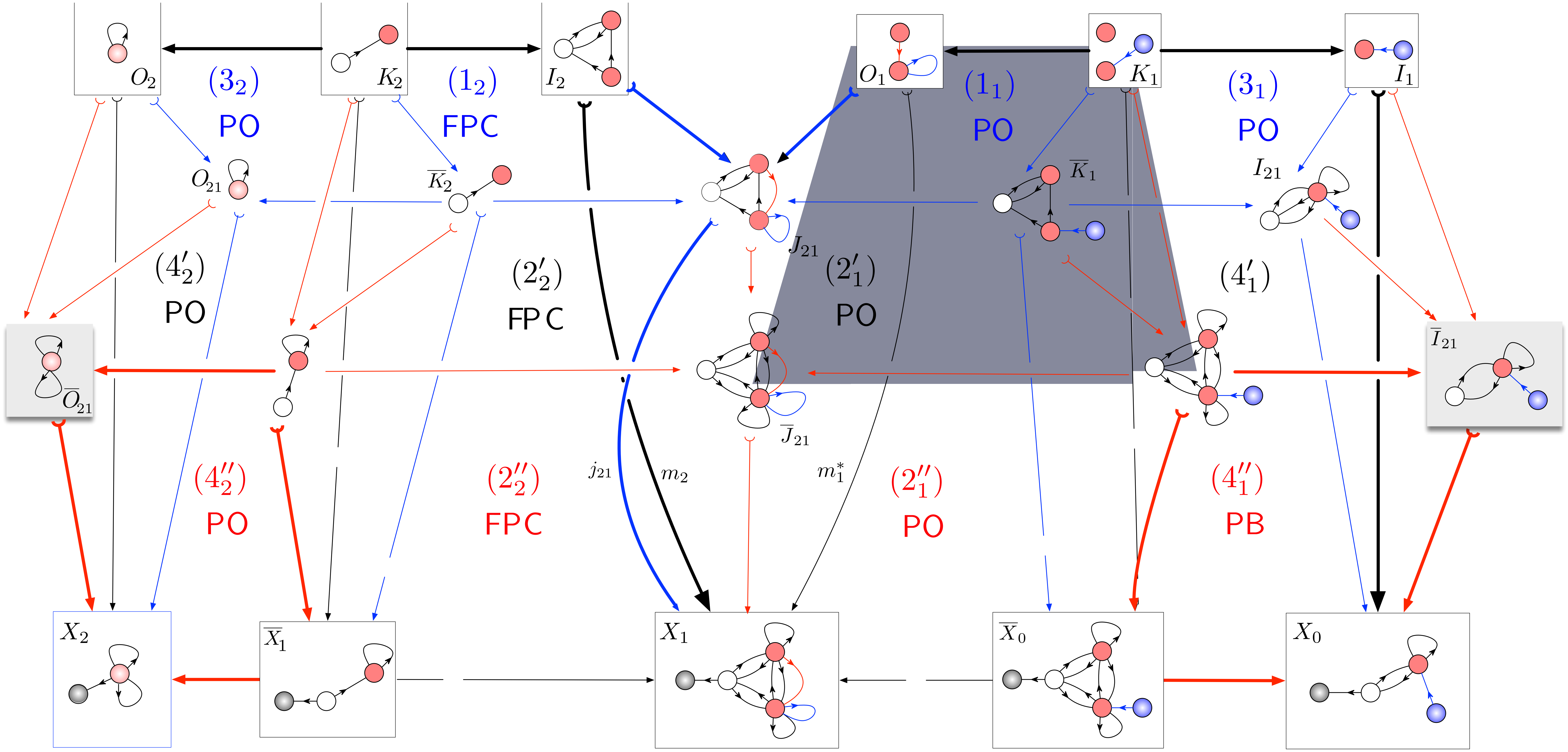

… then take a **pullback**,

# **Concurrent rule composition** for **non-linear SqPO rewriting**



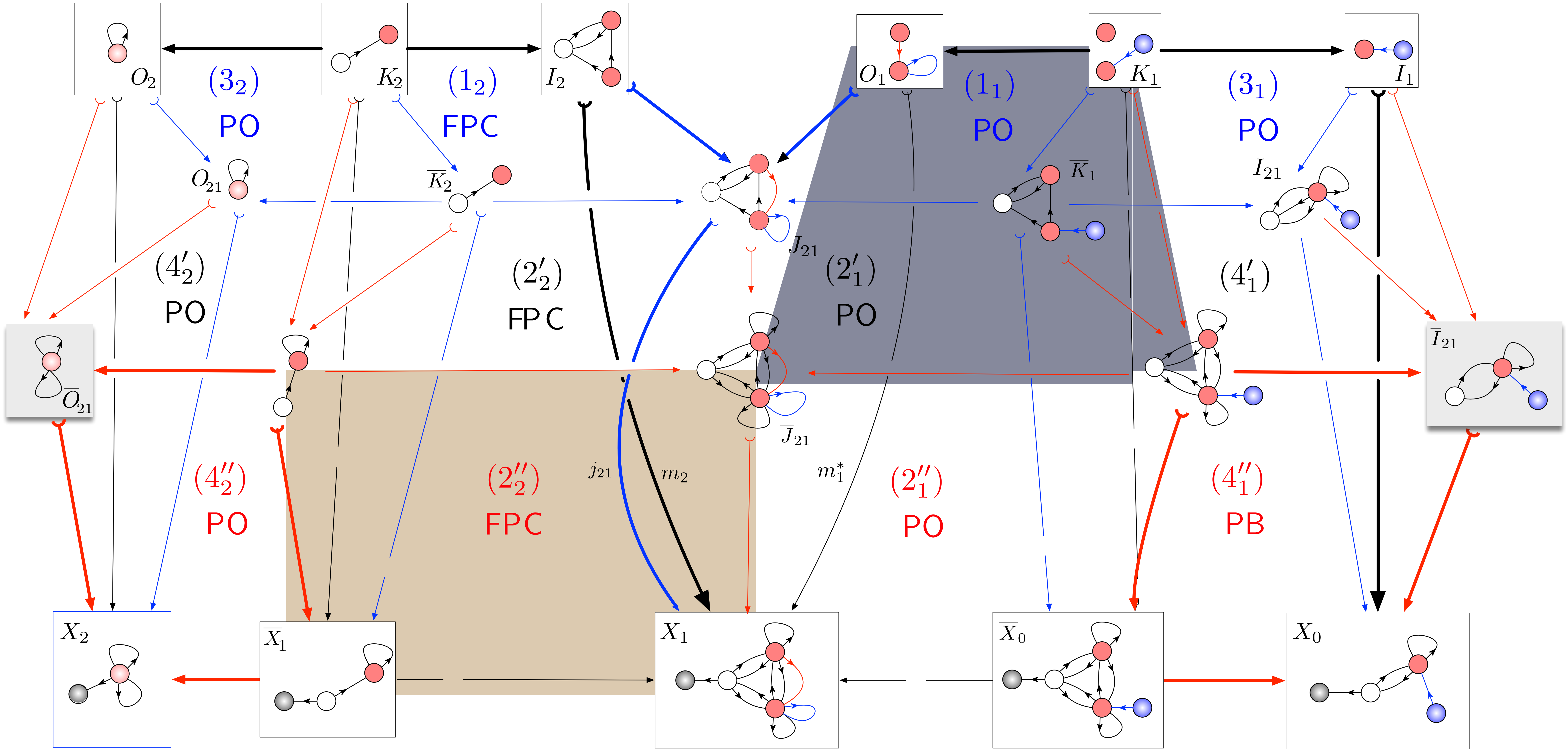

… then take a **pullback**, resulting in **two FPC squares**.

# Concurrent rule composition for non-linear SqPO rewriting



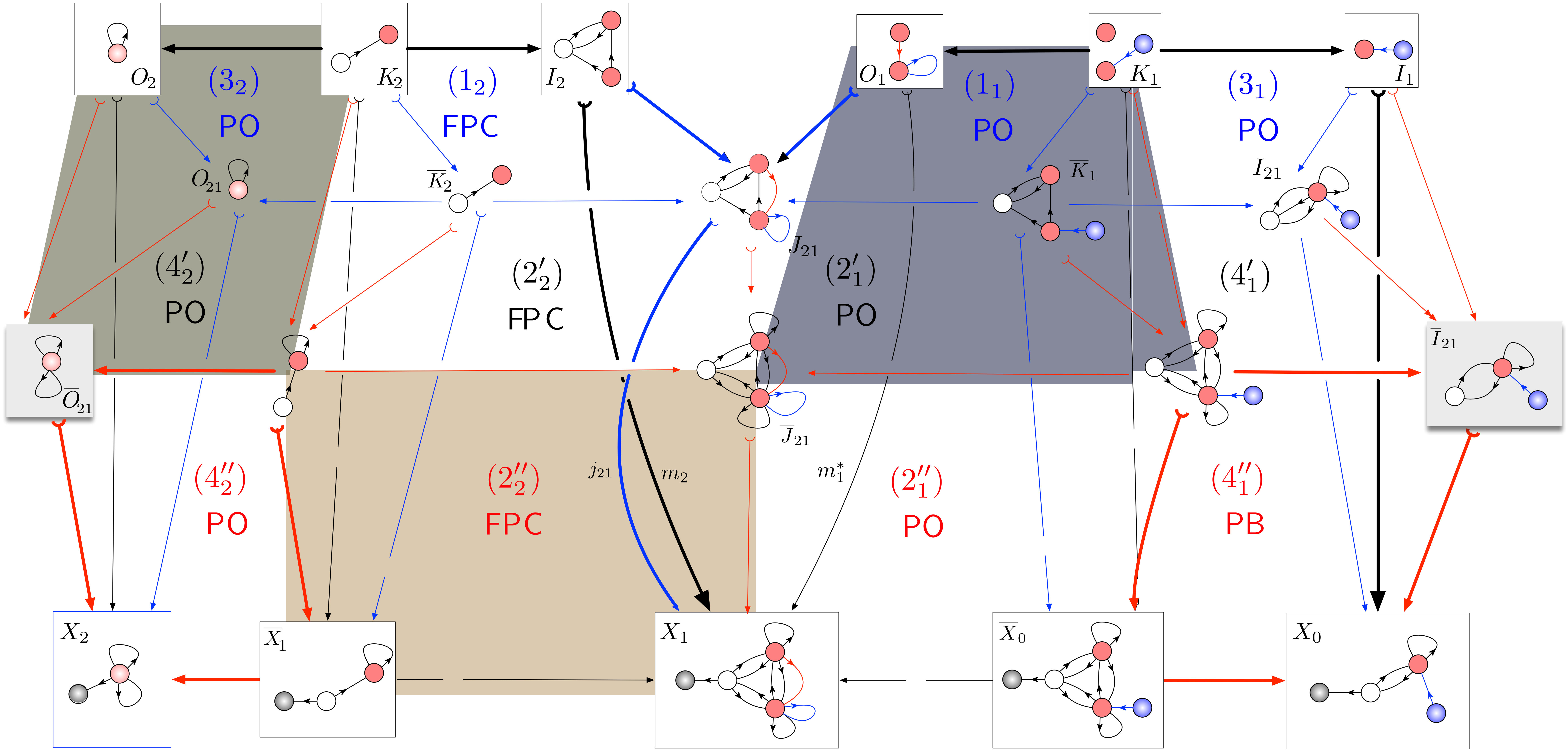

… then take a **pushout**,

# Concurrent rule composition for non-linear SqPO rewriting



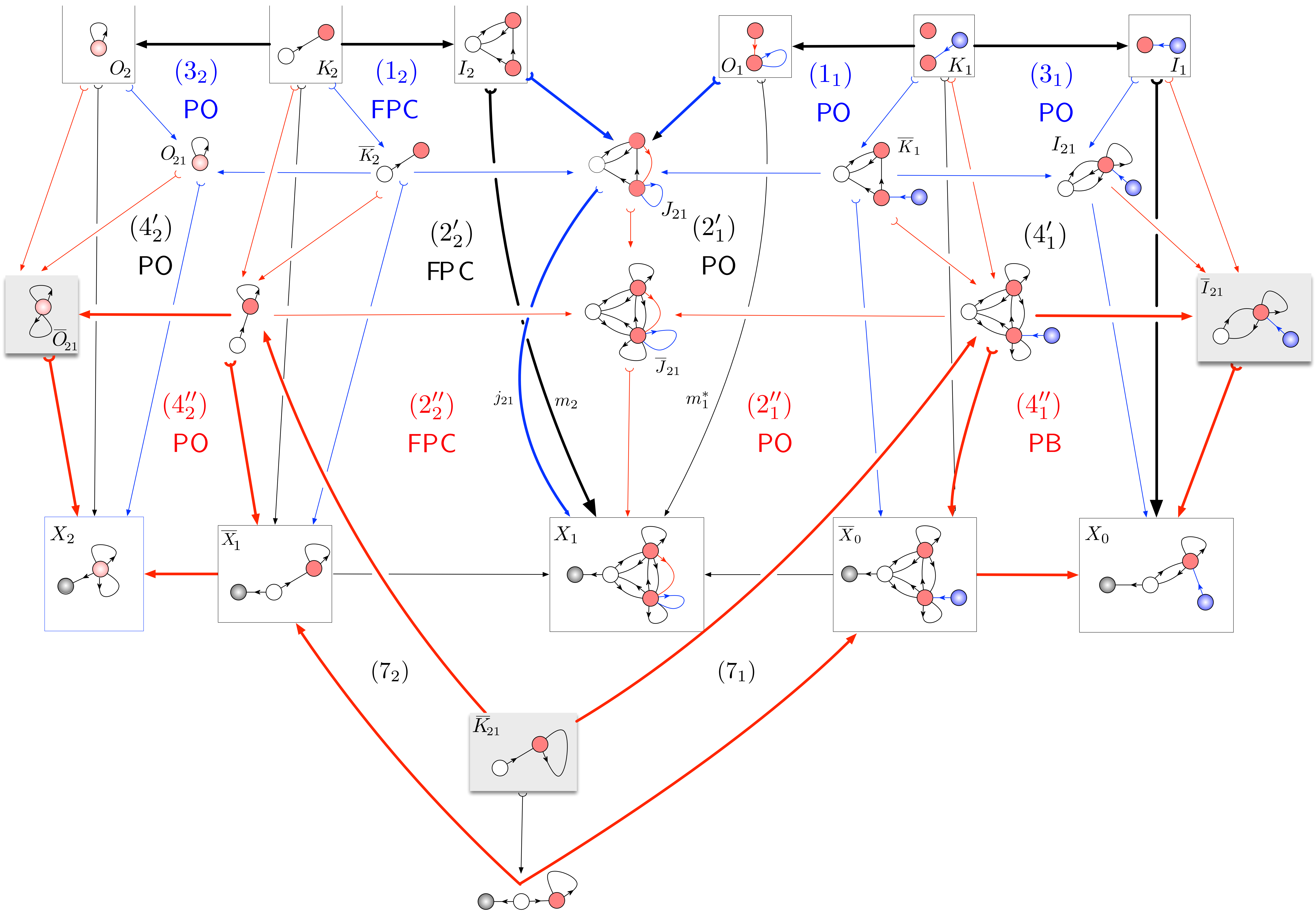

… then take a **pushout**, a **pullback**,

# Concurrent rule composition for non-linear SqPO rewriting

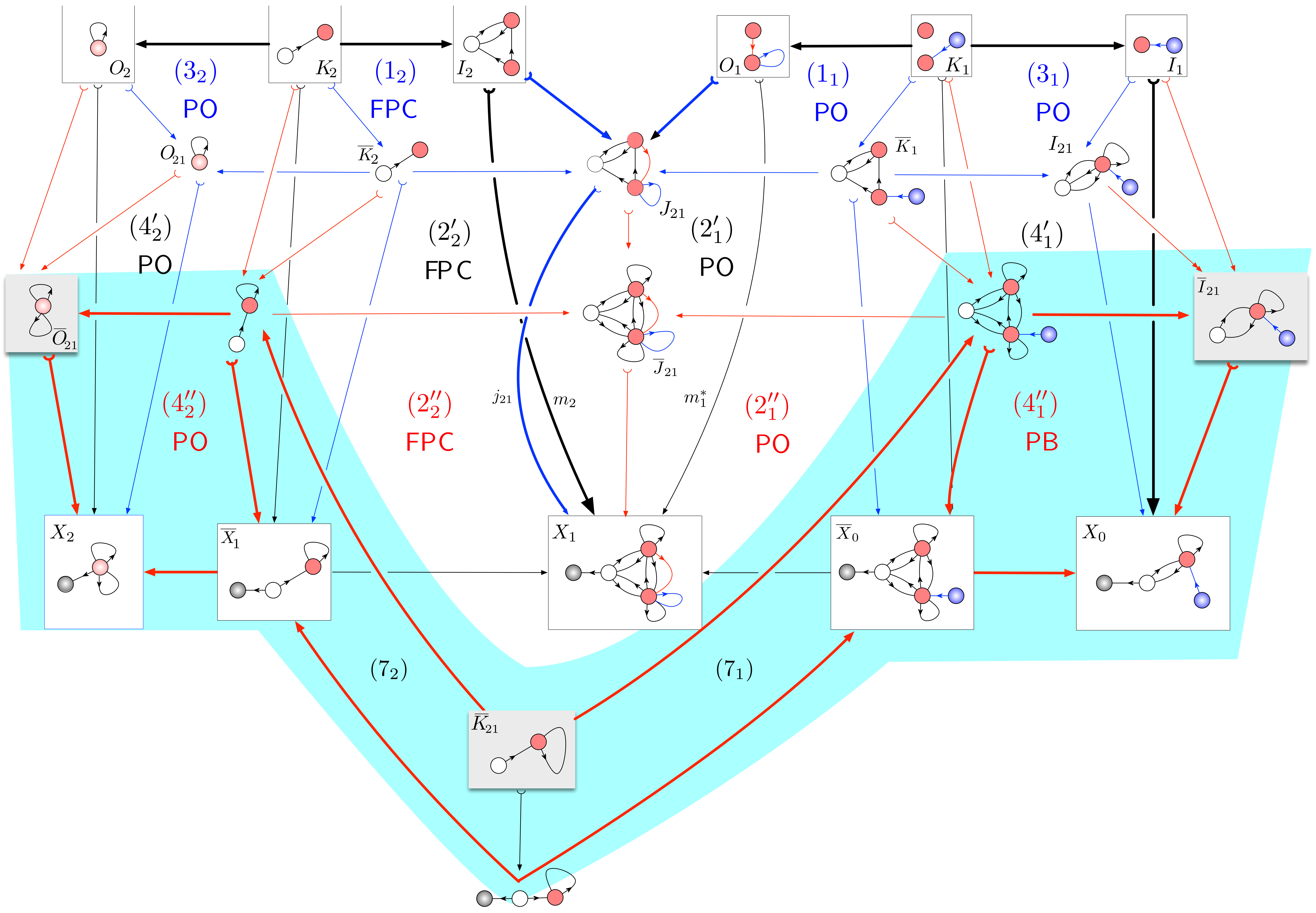



… then take a **pushout**, a **pullback**, and a **pushout**…

# **Concurrent rule composition** for **non-linear SqPO rewriting**



# Concurrent rule composition for non-linear SqPO rewriting



# Concurreny theorem for non-linear SqPO rewriting

Let $\mathbf{C}$ be a quasi-topos, let $X_0 \in \mathsf{obj}(\mathbf{C})$ be an object, and let $r_2, r_1 \in \mathsf{span}(\mathbf{C})$ be two (generic) rewriting rules.

1. Synthesis: For every pair of admissible matches $m_1 \in M_{r_1}^{SqPO}(X_0)$ and $m_2 \in M_{r_2}^{SqPO}(r_{1_{m_1}}(X_0))$, there exist an admissible match $\mu \in \mathcal{M}_{r_2}^{SqPO}(r_1)$ and an admissible match $m_{21} \in M_{r_{21}}^{SqPO}(X_0)$ (for $r_{21}$ the composite of $r_2$ with $r_1$ along $\mu$) such that $r_{21_{m_{21}}}(X_0) \cong r_{2_{m_2}}(r_{1_{m_1}}(X_0))$.
2. Analysis: For every pair of admissible matches $\mu \in \mathcal{M}_{r_2}^{SqPO}(r_1)$ and $m_{21} \in M_{r_{21}}^{SqPO}(X_0)$ (for $r_{21}$ the composite of $r_2$ with $r_1$ along $\mu$), there exists a pair of admissible matches $m_1 \in M_{r_1}^{SqPO}(X_0)$ and $m_2 \in M_{r_2}^{SqPO}(r_{1_{m_1}}(X_0))$ such that $r_{2_{m_2}}(r_{1_{m_1}}(X_0)) \cong r_{21_{m_{21}}}(X_0)$.
3. Compatibility: If in addition $\mathbf{C}$ is finitary, i.e., if for every object of $\mathbf{C}$ there exist only finitely many regular subobjects up to isomorphisms, the sets of pairs of matches $(m_1, m_2)$ and $(\mu, m_{21})$ are isomorphic if they are suitably quotiented by universal isomorphisms, i.e., by universal isomorphisms of $X_1 = r_{1_{m_1}}(X_0)$ and $X_2 = r_{2_{m_2}}(X_1)$ for the set of pairs $(m_1, m_2)$, and by the universal isomorphisms of multi-sums, multi-POCs and FPAs for the set of pairs $(\mu, m_{21})$, respectively.

# Plan of the talk

1. **Quasi-topoi** in rewriting theory
2. **Prerequisites** for non-linear rewriting
3. **Non-linear DPO rewriting**
4. ***Non-linear SqPO rewriting***
5. Conclusion and outlook

# Conclusion

- We have introduced a **new "compositional" theory** for **non-linear DPO- and SqPO-type rewriting** with completely **generic rules** over **quasi-topoi**.
- Somewhat surprisingly, **quasi-topoi** pose a very natural setting for both types of semantics, admitting *without additional axioms* the crucial constructions of **multi-sums**, **multi-pushout complements** and **FPC pushout augmentations**.
- **"Compositionality"** refers to the existence of suitable **concurrency theorems**, which for the case of **DPO** rewriting requires the underlying category to be **regular-mono-adhesive**.

# Outlook

- Investigate the **associativity of rule compositions**, which if it were to hold would permit to formulate **rule algebras** and **tracelets** in order to utilize non-linear rewriting theory for CTMCs, enumerative graph combinatorics, network theory and modeling, ….
- Is it strictly necessary for the case of **non-linear DPO-type rewriting** to be formulated over a **rm-adhesive category**, or could this requirement be relaxed to **quasi-topoi** as in the case of **non-linear SqPO-type rewriting**?

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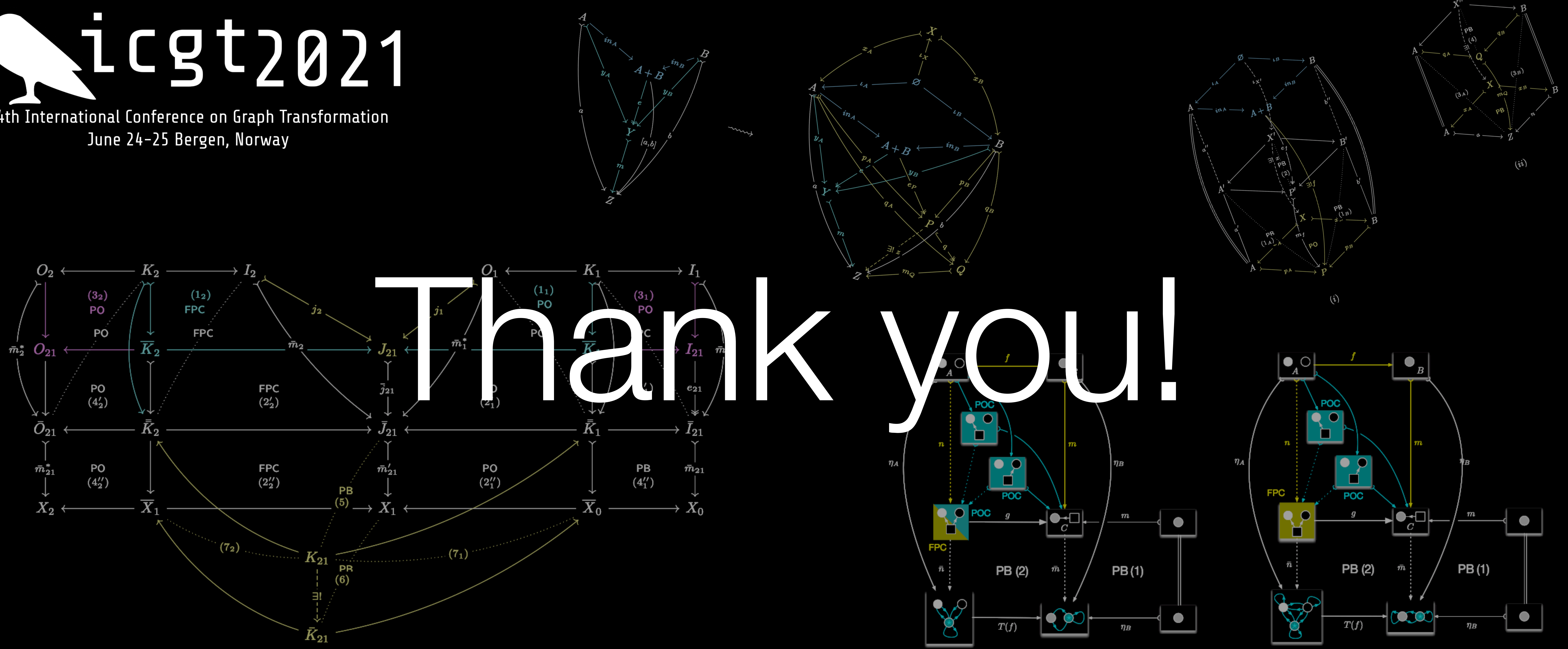

# Thank you!

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

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**Concurrency theorems for non-linear rewriting theories**

*Joint work with **Jean Krivine (IRIF)** and **Russ Harmer (ENS Lyon)***

ICGT'21 (online), June 24, 2021

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