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A hyperdoctrinal concurrency theorem for conditional double-pushout rewriting

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Abstract

We introduce a construction of a labelled transition system induced by any hyperdoctrine, in Lawvere’s sense. Furthermore, we show that, when the considered hyperdoctrine satisfies a generalised form of the well-known Beck-Chevalley condition, its induced labelled transition system satisfies a concurrency theorem that generalises the one from conditional double-pushout rewriting. For any suitable base category, we also show that the syntax and deduction rules for conditions introduced by König et al. form an initial object in a suitable category of hyperdoctrines. Finally, we exploit this to provide a new account of the known link between arrow-based and cospan-based conditions.

2012 ACM Subject Classification Replace ccsgdesc macro with valid one

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1 Introduction

1.1 Motivation: the concurrency theorem

Double-Pushout (DPO) rewriting is a standard categorical framework in graph rewriting. Intuitively, in that setting, a rewriting rule specifies a pattern, and inside that pattern, a part that ought to be replaced by another one when a match is found, i.e., when the pattern is found in a larger context. Conditional DPO rewriting extend DPO rewriting with the possibility of filtering matches, by attaching to rewriting rules a predicate called a [condition](#)³.

The basic setting for conditional DPO rewriting consists of a base category $\mathbb{B}$, typically with [adhesivity](#) properties [12]. One then defines a (syntactic) notion of [condition](#) φ on an object $b \in \mathbb{B}$ [16, 2], whose semantics $\llbracket \varphi \rrbracket$ is given by a set of morphisms from b ⁴. A [conditional rewrite rule](#) is a span

$$x \xleftarrow{f} m \xrightarrow{g} y, \tag{1}$$

equipped with a condition φ on m . Finally, an object $b \in \mathbb{B}$ *rewrites* to some $c \in \mathbb{B}$ through such a rule iff there exists objects and morphisms completing (1) and b, c into two pushouts, as follows,

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³ Highlighted words are hyperlinks to their definitions, thanks to Colcombet’s **knowledge** package.

⁴ Actually, the semantics is more general; we stick to sets of morphisms in this introduction for simplicity.

$$\begin{array}{ccccc}
x & \xleftarrow{f} & m & \xrightarrow{g} & y \\
h \downarrow & & \downarrow k & & \downarrow l \\
b & \xleftarrow{f'} & z & \xrightarrow{g'} & c
\end{array}$$

such that $k \in \llbracket \varphi \rrbracket$.

► **Remark 1.** Sometimes, l , k , and h are required to belong to some well-behaved class $\mathbb{M}$ of monos: this is called the **input-linear** setting. Additionally, f and g are sometimes also required to belong to $\mathbb{M}$: this is called the **linear** setting.

Rewrites form a transition system, in the following “typed” sense:

► **Definition 2.** A **transition system labelled** in some *(directed, multi-)graph* G is a graph morphism $G \rightarrow \mathbf{Rel}$ to the graph underlying the category $\mathbf{Rel}$ of sets and relations.

► **Remark 3.** For a labelled transition system $F: G \rightarrow \mathbf{Rel}$, the set $F(a)$ associated to any vertex $a \in G$ is thought of as a set of states of type a , while the relation $F(e): F(a) \rightarrow F(b)$ associated to any edge $e \in G(a, b)$ is thought of as the set of e -labelled transitions.

► **Notation 1.** We denote by $x \xRightarrow{e} y$ the fact that x and y are related by $F(e)$, when F is clear from context.

► **Example 4.** The **labelled transition system** of conditional DPO rewrites has as underlying graph the one with objects of $\mathbb{B}$ as vertices, and conditional rules $\rho = (x, m, y, f, g, \varphi)$ as edges $a \rightarrow b$. A state is an object of $\mathbb{B}$, and we have $b \xRightarrow{\rho} c$ when b rewrites to c through ρ .

A cornerstone result in conditional DPO rewriting is the concurrency theorem, which may be stated as follows.

► **Definition 5.** A **labelled transition system** $F: G \rightarrow \mathbf{Rel}$ is **concurrency-friendly** iff for any consecutive edges $a \xrightarrow{\rho_1} b \xrightarrow{\rho_2} c$ in G , there exists a set $(\rho_2 \circ \rho_1) \subseteq G(a, c)$ of “composite” edges characterising two-step transitions through ρ_1 and ρ_2 in the sense that, for all states $x \in F(a)$ and $y \in F(c)$, the following are equivalent

- (a) there is a two-step transition $x \xRightarrow{\rho_1} z \xRightarrow{\rho_2} y$ for some z , and
- (b) there is a transition $x \xRightarrow{\rho} y$ for some $\rho \in \rho_2 \circ \rho_1$.

► **Theorem 6** (Informal concurrency theorem). Under suitable hypotheses, the **labelled transition system** of conditional DPO rewriting is concurrency-friendly.

Concurrency theorems have been proved at increasing levels of generality [5, 7, 6, 1], with rather intricate axiomatisations and proofs.

1.2 Contribution

The main contribution of this paper is a generalisation of the concurrency theorem from the specific setting of conditional DPO rewriting to Lawvere’s **hyperdoctrines** [15, 14]. We show that any hyperdoctrine generates a transition system, which under mild hypotheses is concurrency-friendly (Theorem 19).

Furthermore, we show that conditional DPO rewriting may be recovered as a class of instances. Specifically, we first recover (Proposition 41) a generalised transition system in which rules are decorated with (semantic) predicates on morphisms (i.e., suitable sets of

morphisms), instead of (syntactic) conditions. In passing, we introduce (Corollary 45) a finer variant of the concurrency theorem which bounds the cardinality of composite rules, which is hopefully useful in a computational setting.

In §4, we then adjust the above, entirely semantic presentation of the concurrency theorem to encompass the syntactic one [2]. For this, we recast the syntactic presentation of conditions as an initial hyperdoctrine in a suitable category (Theorem 67), and its semantics as the morphism induced by initiality (Proposition 70). We then exploit this to recover the [labelled transition system](#) of conditional DPO rewriting, by a suitable base change (Proposition 71).

Finally, we relate our mainly cospan-based presentation of conditions and their semantics to another standard presentation based on mere morphisms. To do this, we show in §5 that any [arrow-based](#) condition φ , suitably defined, induces a cospan-based conditions φ' , such that $m \in \llbracket \varphi \rrbracket$ iff the cospan $\cdot \xrightarrow{m} \cdot \xleftarrow{!} \emptyset$ is in $\llbracket \varphi' \rrbracket$.

1.3 Related work

The most general version of our results amounts to a version of the “non-linear” DPO rewriting concurrency theorem introduced by Behr et al. [1] extended with conditions. As such, it constitutes a novel semantics not previously systematically studied (if, perhaps, foreseen by Bruggink et al. [2]) in the rewriting literature.

Furthermore, the graphs underlying concurrency-friendly [labelled transition systems](#) or [transformers](#) feature some relational form of composition, hence appear quite close to the catoids of Fahrenberg et al. [8]. We also have a related quantaloid-based version of the concurrency theorem, which we omitted for lack of space. We leave for further work the investigation of links between both approaches.

Finally, we recently found out that our generalised Beck-Chevalley condition seems close to a decategorified version of Caramello’s “relative” Beck-Chevalley condition [3, Definition 5.1(e)(i)].

1.4 Plan

After a detailed overview in §2, we introduce conditionality and the concurrency theorems in §3. We then deal with syntactic conditions in §4 and with [arrow-based](#) conditions in §5. Finally, we conclude in §6.

2 Overview

Before diving into technical detail, let us informally explain our statement of the concurrency theorem (§3), and its link with usual presentations (Sections 4–5).

► **Remark 7.** Let us start with a caveat: the above version of conditional DPO has objects of $\mathbb{B}$ as states. We actually recover a version where states are isomorphism-classes of objects. We consider this as technical detail.

2.1 Hyperdoctrinal concurrency

As announced above, our first goal is to recover [semantic](#) conditional DPO rewriting, i.e., the version where rules are decorated with sets of morphisms instead of conditions, as an instance of some construction on hyperdoctrines.

For this, let us briefly recall:

► **Definition 8.** A (complete boolean) *hyperdoctrine* on a category $\mathbb{C}$ is a functor $P: \mathbb{C} \rightarrow \mathbf{CBAAlg}$ to the category of complete boolean algebras and bicontinuous maps.

► **Notation 2.** Because bicontinuous maps between sup-lattices have left and right adjoints, each $P(f)$, also denoted by f^* , has adjoints $\exists_f \dashv f^* \dashv \forall_f$.

Here is what will be our favourite source of hyperdoctrines:

► **Definition 9.** For any locally small category $\mathbb{C}$ and distinguished object Z , the *subhom hyperdoctrine* $\mathcal{P}_Z$ is the composite $\mathbb{C} \xrightarrow{yz} \mathbf{Set}^{op} \xrightarrow{\mathcal{P}} \mathbf{CBAAlg}$.

► **Example 10.** For any category $\mathbb{B}$ with suitable adhesivity properties⁵ and an initial object $\emptyset$, we consider the relevant category $\mathbb{C}$ of cospans, e.g., linear or input-linear, and then take the subhom hyperdoctrine $\mathcal{P}_\emptyset$. Let us observe:

- Each $\mathcal{P}_\emptyset(a)$ consists of sets of morphisms $a \rightarrow \emptyset$ in $\mathbb{C}$, i.e., sets of isomorphism classes of cospans $a \rightarrow b \leftarrow \emptyset$, which we take as the notion of predicate for *semantic* conditional DPO rewriting. In particular, this includes interpretations of conditions.
- Furthermore, $\mathcal{P}_\emptyset(\emptyset)$ consists of isomorphism-classes of objects, which are states of the semantic conditional DPO transition system.
- Finally, semantic *conditional rules* are equivalent to particular cospans $\emptyset \xrightarrow{f^-} m \xleftarrow{g^-} \emptyset$ in $\mathbb{C}$, equipped with a predicate $\varphi \in \mathcal{P}_\emptyset(m)$, where k^- denotes the cospan $\emptyset \xrightarrow{!} b \xleftarrow{k} a$, for any $k \in \mathbb{B}(a, b)$.

The relevance of this example for *semantic* conditional DPO rewriting lies in the fact that we can characterise transitions in terms of the logic of the hyperdoctrine $\mathcal{P}_\emptyset$. Indeed, for all *semantic* conditional rules ρ given by some cospan (1) and predicates $\varphi \in \mathcal{P}_\emptyset(m)$, one can show that we have

$$b \xRightarrow{\rho} c \quad \Longleftrightarrow \quad c \in \exists_{g^-}(\varphi \wedge (f^-)^*\{b\}),$$

where on the right we implicitly coerce objects $x \in \{b, c\}$ into (isomorphism-classes of) cospans $\emptyset \xrightarrow{!} x \xleftarrow{!} \emptyset$.

We may thus view *semantic* conditional DPO rewriting as an instance of a generic *labelled transition system*, induced by any hyperdoctrine as follows.

► **Definition 11.** Consider any hyperdoctrine $P: \mathbb{C} \rightarrow \mathbf{CBAAlg}$.

- A *P-rule* ρ consists of a cospan $a \xrightarrow{f} m \xleftarrow{g} b$ in $\mathbb{C}$ and a predicate $\varphi \in P(m)$.
- Let $\text{Cospan}(P)$ denote the *graph* with objects of $\mathbb{C}$ as vertices, and *P-rules* ρ as above as edges $a \rightarrow b$.
- Finally, we define the *Cospan(P)-labelled transition system* induced by P as follows:
 - states consist of an object $a \in \mathbb{C}$, equipped with an atom x of $P(a)$; and
 - we have $x \xRightarrow{\rho} y$ iff $y \leq \exists_g(\varphi \wedge f^*(x))$.

► **Example 12.** In the setting of Example 10, taking $P = \mathcal{P}_\emptyset$ and restricting to *P-rules* with $a = b = \emptyset$, we recover exactly the transition system of *semantic* conditional DPO rewriting.

In fact, this labelled transition system may be derived from what we call a *transformer*⁶, a notion that does not rely on atoms of the hyperdoctrine and which we find more convenient

⁵ For the expert reader: $\mathbb{B}$ is $\mathbb{M}$ -adhesive and $\emptyset$ is $\mathbb{M}$ -initial, see §3.2.

⁶ Transformers are here meant to evoke Hoare semantics [11, 4].

to work with. We thus prefer to state and prove the concurrency theorem at the level of transformers, and derive the version for labelled transition systems as a corollary. More precisely:

► **Definition 13.** For any graph G , a *G -transformer* is a graph morphism $G \rightarrow \mathbf{Sup}$ to the graph underlying the category $\mathbf{Sup}$ of sup-lattices and cocontinuous maps.

► **Notation 3.** We let $\rho \cdot X := F(\rho)(X)$, for any $F: G \rightarrow \mathbf{Sup}$, $\rho \in G(a, b)$, and $X \in F(a)$, whenever F is clear from context.

► **Definition 14.** For any G -transformer $F: G \rightarrow \mathbf{Sup}$, we define its associated *labelled transition system* $\tilde{F}: G \rightarrow \mathbf{Rel}$ by letting $\tilde{F}(a)$ be the set of atoms of $F(a)$, and

$$x \xRightarrow{\rho} y \quad \text{iff} \quad y \leq \rho \cdot x.$$

As planned, we show that the labelled transition system of Definition 11 may be recovered from a suitable transformer.

► **Definition 15.** Let

$$\begin{aligned} act_P: \text{Cospan}(P) &\rightarrow \mathbf{Sup} \\ a &\mapsto P(a) \end{aligned}$$

$$act_P(\rho)(X) = \exists_g(\varphi \wedge f^*(X)), \quad \text{for } \rho = (a \xrightarrow{f} m \xleftarrow{g} b, \varphi \in P(m)) \text{ and } X \in P(a).$$

► **Proposition 16.** The $\text{Cospan}(P)$ -labelled transition system of Definition 11 is precisely $\widetilde{act_P}$.

Semantic conditional DPO rewriting is thus an instance of a generic construction on hyperdoctrines. Let us now introduce concurrency-friendliness for transformers, and relate it to the labelled transition systems version.

► **Definition 17.** A transformer $F: G \rightarrow \mathbf{Sup}$ is *concurrency-friendly* if, for any consecutive edges $a \xrightarrow{\rho_1} b \xrightarrow{\rho_2} c$ and $X \in F(a)$, there is a “composite” set $\rho_2 \circ \rho_1$ of edges $a \xrightarrow{\rho} c$ such that

$$F(\rho_2) \circ F(\rho_1) = \bigvee_{\rho \in \rho_2 \circ \rho_1} F(\rho),$$

where the sup is taken pointwise.

► **Proposition 18.** For any concurrency-friendly transformer $F: G \rightarrow \mathbf{Sup}$ such that each $F(a)$ is atomic distributive (e.g., a power set), the induced labelled transition system $\tilde{F}$ is concurrency-friendly.

Our goal is now to investigate when the transformer act_P is concurrency-friendly. When the base category $\mathbb{C}$ has pushouts and P satisfies the standard Beck-Chevalley condition⁷, we can take $\rho_2 \circ \rho_1$ to be the singleton given by pushout, and transports both conditions there.

However, the cospan categories encountered in the wild world of conditional DPO rewriting rarely have pushouts. Instead, they have what Bruggink et al. [2] call “representative

⁷ The Beck-Chevalley condition is generally stated in terms of contravariant hyperdoctrines and pullbacks; we use covariant hyperdoctrines and pushouts, just to save a bunch of $-^{op}$ s.

squares”, which we rename to **weak multipushouts**. (In fact, all small categories have weak multipushouts, see Proposition 30.)

So our solution to cover all desired applications consists in **generalising the Beck-Chevalley condition** from pushouts to weak multipushouts (Definition 27). We then call **conditional** any hyperdoctrine satisfying the **generalised Beck-Chevalley condition**, and obtain:

► **Theorem 19** (Hyperdoctrinal concurrency). *The **transformer** act_P induced by any **conditional hyperdoctrine** P on a small category is **concurrency-friendly**.*

► **Remark 20**. In fact, we introduce a more general notion of **K -conditional hyperdoctrine**, for a choice $K(f, g)$ of **multisquare** for each span $a \xleftarrow{f} m \xrightarrow{g} b$. We then show (Lemma 28) that act_P is **concurrency-friendly** for a notion of rule composition derived from K .

► **Corollary 21**. *The **labelled transition system** induced by any conditional hyperdoctrine of the shape $\mathbb{C} \rightarrow \mathbf{Set}^{op} \xrightarrow{P} \mathbf{CBA}lg$ with $\mathbb{C}$ small is **concurrency-friendly**.*

Proof. By Proposition 18. ◀

As announced in §1.2, we propose a less trivial variant which may be useful for computational purposes:

► **Definition 22**. A **transformer** $F: G \rightarrow \mathbf{Sup}$ is **finitely concurrency-friendly** iff it is **concurrency-friendly** for a choice of finite composite sets $\rho_2 \circ \rho_1$.

► **Theorem 23** (Finitary hyperdoctrinal concurrency). *The **transformer** act_P induced by any **conditional hyperdoctrine** P on a category with finite **weak multipushouts** is **finitely concurrency-friendly**.*

In order to demonstrate that this refined version is widely applicable, we provide (Theorem 44) a generic construction of finite weak multipushouts based on a variant of Gabriel et al.’s finitary adhesive categories [9]. More precisely, we show (Theorem 44) that, for any sufficiently **well-behaved** category $\mathbb{B}$ and orthogonal factorisation system (E, M) on it, the category $\mathbf{Cospan}(M, O)$ has finite weak multipushouts, where $O \in \{M, B\}$. (Having this flexibility on O allows us to cover both **linear** and **input-linear** cospans.)

2.2 Syntactic conditions

Our next goal is to recover concurrency friendliness for the standard presentation of conditional DPO rewriting, featuring **rules** decorated with syntactic conditions.

Here, there is a size issue which we must take care of. Indeed, conditions feature arbitrary sups and infs, which matches our choice of complete boolean algebras defining hyperdoctrines. However, the conjunction of “initiality” and “complete boolean algebra” should trigger a warning, as it is well known that the free complete boolean algebra on a given set is a subtle object. Indeed, unless one restricts to the atomic case, it is *a priori* a proper class.

We see two ways of handling this situation: either embrace the universe bump, or impose size restrictions. Because DPO rewriting is most often applied in programming and verification contexts, we opt for the latter solution.

We thus introduce **λ -hyperdoctrines**, whose fibres are merely required to have λ -small sups and infs, for some fixed, regular⁸ cardinal λ . We then generalise **conditionality** to

⁸ Regularity may be taken to mean that the category $\mathbf{Set}_\lambda$ of sets of cardinality $< \lambda$ is closed under λ -small coproducts in $\mathbf{Set}$.

this setting, and characterise syntactic conditions as an initial conditional λ -hyperdoctrine $\lambda\text{-cond}_{\mathbb{C}}$, for any fixed, small category $\mathbb{C}$ with λ -small weak multipushouts. In particular, we prove that this result applies to categories of the form $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ for [well-behaved](#) $\mathbb{B}$ (Proposition 53).

This at last leads us to a [transformer](#) for conditional DPO rewriting in which rules are decorated with syntactic conditions, but predicates are sets of morphisms – which is necessary to get the right states in the induced labelled transition system (Definition 14). This [transformer](#) is the composite below, where the first morphism is induced by initiality.

$$\mathbf{Cospan}(\lambda\text{-cond}_{\mathbb{C}}) \rightarrow \mathbf{Cospan}(\mathcal{P}_{\emptyset}) \xrightarrow{\text{act}\rho_{\emptyset}} \mathbf{Sup} \quad (2)$$

2.3 Arrow-based conditions

Finally, we deal with arrow-based conditions. The setting consists of a [well-behaved](#) category $\mathbb{B}$ with orthogonal factorisation system $(\mathbb{E}, \mathbb{M})$ and $\mathbb{O} \in \{\mathbb{M}, \mathbb{B}\}$ as above. By definition, [arrow-based](#) conditions are given by the initial [conditional](#) λ -hyperdoctrine $\lambda\text{-cond}_{\mathbb{M}}$ on $\mathbb{M}$, and their semantics is taken in a variant of the [subhom](#) construction, namely the composite

$$\mathbb{M} \hookrightarrow \mathbb{B} \xrightarrow{\coprod_{b \in \mathbb{B}} \gamma_b} \mathbf{Set}^{op} \xrightarrow{\mathcal{P}} \mathbf{CBA} \mathbf{lg}, \quad (3)$$

which we denote by $\mathcal{P}_{\sqcup}$. In order for [arrow-based](#) conditions to have semantics in $\mathcal{P}_{\sqcup}$, we check:

► **Proposition 24.** *The λ -hyperdoctrine $\mathcal{P}_{\sqcup}$ on $\mathbb{M}$ is [conditional](#).*

By initiality, we get a unique morphism $\lambda\text{-cond}_{\mathbb{M}} \rightarrow \mathcal{P}_{\sqcup}$ over $\mathbb{M}$. In order to relate this to the corresponding morphism $\lambda\text{-cond}_{\mathbf{Cospan}(\mathbb{M}, \mathbb{O})} \rightarrow \mathcal{P}_{\mathbb{Z}}$ on the cospan-based side, we show that any functor $F: \mathbb{C} \rightarrow \mathbb{D}$ preserving weak multipushouts between categories with λ -small weak multipushouts induces, by precomposition, a functor $\Delta_F: \lambda\text{-CHDoct}_{\mathbb{D}} \rightarrow \lambda\text{-CHDoct}_{\mathbb{C}}$ between their categories of [conditional](#) λ -hyperdoctrines. This allows us to reindex the unique morphism $\lambda\text{-cond}_{\mathbf{Cospan}(\mathbb{M}, \mathbb{O})} \rightarrow \mathcal{P}_{\mathbb{Z}}$ along the canonical embedding $E^{\mathbb{M}, \mathbb{O}}: \mathbb{M} \rightarrow \mathbf{Cospan}(\mathbb{M}, \mathbb{O})$. We thus obtain a morphism $\Delta_{E^{\mathbb{M}, \mathbb{O}}}(\lambda\text{-cond}_{\mathbf{Cospan}(\mathbb{M}, \mathbb{O})}) \rightarrow \Delta_{E^{\mathbb{M}, \mathbb{O}}}(\mathcal{P}_{\mathbb{Z}})$ over $\mathbb{M}$. We finally exhibit a morphism $\mathcal{P}\alpha^{\mathbb{M}, \mathbb{O}}: \Delta_{E^{\mathbb{M}, \mathbb{O}}}(\mathcal{P}_{\mathbb{Z}}) \rightarrow \mathcal{P}_{\sqcup}$, and deduce from initiality that the composite

$$\lambda\text{-cond}_{\mathbb{M}} \xrightarrow{!} \Delta_{E^{\mathbb{M}, \mathbb{O}}}(\lambda\text{-cond}_{\mathbf{Cospan}(\mathbb{M}, \mathbb{O})}) \rightarrow \Delta_{E^{\mathbb{M}, \mathbb{O}}}(\mathcal{P}_{\mathbb{Z}}) \xrightarrow{\mathcal{P}\alpha^{\mathbb{M}, \mathbb{O}}} \mathcal{P}_{\sqcup}$$

agrees with the semantics of [arrow-based](#) conditions. This unfolds to the fact that any [arrow-based](#) condition $\varphi \in \lambda\text{-cond}_{\mathbb{M}}(b)$ is mapped to some $!\varphi \in \lambda\text{-cond}_{\mathbf{Cospan}(\mathbb{M}, \mathbb{O})}(b)$ such that

$$(b \xrightarrow{f} b') \in \llbracket \varphi \rrbracket \quad \Longleftrightarrow \quad (b \xrightarrow{f} b' \xleftarrow{!} \emptyset) \in \llbracket !\varphi \rrbracket. \quad (4)$$

3 Hyperdoctrinal concurrency

In this section, we flesh out the details of §2.1.

3.1 Conditional hyperdoctrines

In this section, we introduce our generalised Beck-Chevalley condition and conditional hyperdoctrines. We finally show that [subhom](#) hyperdoctrines are conditional.

Let us start by recalling the plain version of Beck-Chevalley.

► **Definition 25.** A hyperdoctrine $P: \mathbb{C} \rightarrow \mathbf{CBA}lg$ satisfies the *Beck-Chevalley* condition on a square as below left,

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ g \downarrow & & \downarrow u \\ c & \xrightarrow[v]{} & d \end{array} \qquad \begin{array}{ccc} P(a) & \xleftarrow{\exists_f} & P(b) \\ g^* \downarrow & \geq & \downarrow u^* \\ P(c) & \xleftarrow[\exists_v]{} & P(d) \end{array}$$

when the canonical inequality $\exists_v \circ u^* \leq g^* \circ \exists_f$ (as above right) is actually an equality.

In the literature, Beck-Chevalley is most often required for pushouts⁷, or for more specific kinds of squares. Here, we move from single squares to what we call *multisquares*:

► **Definition 26.** A *multisquare* on a span $b \xleftarrow{f} a \xrightarrow{g} c$ in a category $\mathbb{C}$ is a set of cospans $b \xrightarrow{u} d \xleftarrow{v} c$ making the square above left commute.

Let us now introduce conditional hyperdoctrines and the generalised Beck-Chevalley condition:

► **Definition 27.** Let $\mathbb{C}$ be a category equipped with a *multisquare* $K(f, g)$ for each span $b \xleftarrow{f} a \xrightarrow{g} c$. A hyperdoctrine $P: \mathbb{C} \rightarrow \mathbf{CBA}lg$ is *K-conditional* when, for any span (f, g) as above, the canonical inequality

$$\bigvee_{(u,v) \in K(f,g)} (\exists_v \circ u^*) \leq g^* \circ \exists_f$$

is in fact an equality, which we call the *generalised Beck-Chevalley* condition.

We are now in a position to state the key lemma towards Theorem 19 and its finitary version Theorem 23:

► **Lemma 28.** Let $\mathbb{C}$ denote any category equipped with a choice $K(f, g)$ of *multisquare* for each span $b \xleftarrow{f} a \xrightarrow{g} c$. Then, for any *K-conditional* hyperdoctrine P on $\mathbb{C}$, the *transformer* act_P is *concurrency-friendly*. The composite of any two *P-rules* given by

$$\left(a \xrightarrow{l} m \xleftarrow{r} b, \quad \varphi \in P(m) \right) \quad \text{and} \quad \left(b \xrightarrow{l'} n \xleftarrow{r'} c, \quad \psi \in P(n) \right),$$

is the set of rules of the form

$$\left(a \xrightarrow{l} m \xrightarrow{u} p \xleftarrow{v} n \xleftarrow{r'} c, \quad u^*(\varphi) \wedge v^*(\psi) \in P(p) \right),$$

where $(u, v) \in K(r, l')$. In particular, if each $K(f, g)$ is finite, then act_P is finitely concurrency-friendly.

We end this subsection by studying the special case where K is a choice of *weak multipushouts*, which we now introduce.

► **Definition 29.** A *weak multipushout* of the span $b \xleftarrow{f} a \xrightarrow{g} c$ is a *multisquare* K which is *jointly weakly initial*, in the sense that, for any cospan $b \xrightarrow{h} x \xleftarrow{k} c$ completing the square, there is a cospan $b \xrightarrow{u} d \xleftarrow{v} c$ in K and a morphism $m: d \rightarrow x$ making the diagram below commute.

$$\begin{array}{ccccc} a & \xrightarrow{g} & c & & \\ f \downarrow & & \downarrow v & \searrow k & \\ b & \xrightarrow{u} & d & \xrightarrow{m} & x \\ & \searrow h & & & \end{array} \tag{5}$$

In this paper, we mostly consider small categories, which trivialise weak multipushouts:

► **Proposition 30.** *Any small category has weak multipushouts.*

Proof. Take all cospans completing the given span. ◀

Let us first observe:

► **Proposition 31.** *Given choices K and L of weak multipushouts on a category $\mathbb{C}$, a hyperdoctrine $P: \mathbb{C} \rightarrow \mathbf{CBA}lg$ is K -conditional iff it is L -conditional.*

► **Definition 32.** *Given a category $\mathbb{C}$ with weak multipushouts, we call a hyperdoctrine on $\mathbb{C}$ **conditional** if it is K -conditional for some choice K of weak multipushouts in $\mathbb{C}$.*

Theorem 19 and Theorem 23 now make rigorous sense, and readily follow from Lemma 28.

Finally, let us show that they apply to the **subhom** hyperdoctrine:

► **Proposition 33.** *For any locally small category $\mathbb{C}$ with weak multipushouts and distinguished object Z , the **subhom** hyperdoctrine $\mathcal{P}_Z$ is conditional.*

Proposition 33 follows easily from the next lemma.

► **Lemma 34.** *For any locally small category $\mathbb{C}$, $Z \in \mathbb{C}$, $f \in \mathbb{C}(a, b)$, $\varphi \in \mathcal{P}_Z(a)$, and $\psi \in \mathcal{P}_Z(b)$, we have:*

- $f^*(\varphi)$ is the set of extensions of morphisms in φ along f ;
- $\exists_f(\psi)$ is the set of composites of f with some morphism in ψ , or equivalently the set of morphisms $a \rightarrow Z$ admitting an extension along f that is in ψ ;
- $\forall_f(\psi)$ is the set of morphisms $a \rightarrow Z$ whose extensions along f are all in ψ .

In symbols:

$$\begin{aligned} f^*(\varphi) &= \mathbb{C}(f, Z)^{-1}(\varphi) &&= \{k \in \mathbb{C}(b, Z) \mid k \circ f \in \varphi\} \\ \exists_f(\psi) &= \mathbb{C}(f, Z)(\psi) &&= \{h \in \mathbb{C}(a, Z) \mid \exists k \in \mathbb{C}(b, Z), h = k \circ f \wedge k \in \psi\} \\ \forall_f(\psi) & &&= \{h \in \mathbb{C}(a, Z) \mid \forall k \in \mathbb{C}(b, Z), h = k \circ f \implies k \in \psi\}. \end{aligned}$$

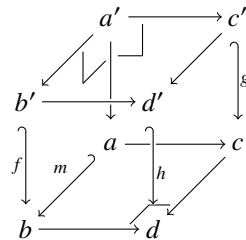
3.2 Covering semantic conditional DPO rewriting

Let us now substantiate the claims of §2.1, and notably Example 12, that the hyperdoctrinal setting covers semantic conditional DPO rewriting. For this, let us recall the standard notion of $\mathbb{M}$ -adhesivity and both related cospan constructions.

► **Definition 35** (Ehrig et al. [5, Definition 2.3]). *Let $\mathbb{M}$ be a wide, replete subcategory of a finitely complete category $\mathbb{B}$, which*

- *consists only of monomorphisms,*
- *is left-cancellative, in the sense that $gf \in \mathbb{M}$ and $g \in \mathbb{M}$ entail $f \in \mathbb{M}$, and*
- *is closed under pullbacks and pushouts along arbitrary morphisms.*

We say that $\mathbb{B}$ is $\mathbb{M}$ -adhesive when it has pullbacks and pushouts along morphisms in $\mathbb{M}$, and furthermore, for any commuting cube as below



if $m, f, g, h \in \mathbb{M}$, the bottom face is a pushout, and both back faces are pullbacks, then the top face is a pushout iff both front faces are pullbacks.

► **Example 36.** Any skeleton $\mathbf{Gph}_f$ of the category of finite graphs is small and $\mathbb{M}$ -adhesive, for $\mathbb{M}$ the class of monos.

► **Definition 37.** For any $\mathbb{M}$ -adhesive category $\mathbb{B}$ and $\emptyset \in \{\mathbb{M}, \mathbb{B}\}$, let $\mathbf{Cospan}(\mathbb{M}, \emptyset)$ denote the category with the same objects as $\mathbb{B}$, and morphisms $a \rightarrow b$ given by isomorphism-classes of cospans $a \xrightarrow{l} x \xleftarrow{r} b$ with $l \in \mathbb{M}$ and $r \in \emptyset$.

► **Remark 38.** The construction is well defined in both cases by definition of $\mathbb{M}$ -adhesivity.

In order to perform our constructions, we furthermore need:

► **Definition 39.** Given a class $\mathbb{M}$ of morphisms in a category $\mathbb{B}$, an object Z of $\mathbb{B}$ is $\mathbb{M}$ -initial iff it is initial in $\mathbb{B}$ and every morphism $Z \rightarrow b$ is in $\mathbb{M}$.

► **Example 40.** By the $\mathcal{P}_\emptyset$ evoked informally in §2.1, and notably in Example 10, we meant more formally the **subhom** $\mathcal{P}_\emptyset$ on any category of the form $\mathbf{Cospan}(\mathbb{M}, \emptyset)$, for any small, $\mathbb{M}$ -adhesive category $\mathbb{B}$ with an $\mathbb{M}$ -initial object $\emptyset$, and $\emptyset \in \{\mathbb{M}, \mathbb{B}\}$. In particular, $\mathbf{Cospan}(\mathbb{M}, \emptyset)$ is small, hence has weak multipushouts by Proposition 30, so $\mathcal{P}_\emptyset$ is indeed **conditional** by Proposition 33.

Furthermore, let us explain why the transition system $\widetilde{\text{act}}_{\mathcal{P}_\emptyset}$ is correct, or in other words justify Example 12:

► **Proposition 41.** For any small, $\mathbb{M}$ -adhesive category $\mathbb{B}$ with $\mathbb{M}$ -initial object $\emptyset$, $\emptyset \in \{\mathbb{M}, \mathbb{B}\}$, and **rule** ρ given by $a \xrightarrow{f} m \xrightarrow{g} b$ in $\mathbf{Cospan}(\mathbb{M}, \emptyset)$ and $\varphi \in \mathcal{P}_\emptyset(m)$ we have

$$x \xRightarrow{\rho} y \quad \Longleftrightarrow \quad y \in \exists_g(\varphi \wedge (f)^*\{x\}).$$

Proof. This is obvious from unfolding the right-hand side using Lemma 34. Indeed, $\varphi \wedge (f)^*\{x\}$ denotes the set of morphisms $h: m \rightarrow \emptyset$ such that $h \in \varphi$ and $\varphi \circ f = x$, i.e., the set of extensions of x (viewed as a morphism $\emptyset \rightarrow \emptyset$ in $\mathbf{Cospan}(\mathbb{M}, \emptyset)$) along f satisfying φ . This further unfolds to (isomorphism-classes of) cospans $m \xrightarrow{h^l} c \xleftarrow{!} \emptyset$ in $\mathbf{Cospan}(\mathbb{M}, \emptyset)$ making the left-hand square below into a pushout. Finally, applying $\exists_g$ unfolds to composing all such cospans with g , thus yielding a pushout as below right.

$$\begin{array}{ccccc}
 \emptyset & \xrightarrow{f^l} & a & \xleftarrow{f^r} & m & \xrightarrow{g^r} & b & \xleftarrow{g^l} & \emptyset \\
 & \searrow ! & \downarrow & \lrcorner & h^l \downarrow & \lrcorner & \downarrow & \swarrow ! & \\
 & & y & \xleftarrow{\quad} & c & \xrightarrow{\quad} & y & & \\
 & & & \nwarrow ! & \uparrow ! & \nearrow ! & & & \\
 & & & & \emptyset & & & &
 \end{array}$$

3.3 Tracking finiteness

In this section, we work on the “finitary” version of the concurrency theorem (Theorem 23). The proof is immediate from Lemma 28, but its applicability is less trivial. We thus exhibit a class of use cases:

► **Definition 42.** Let $\mathbb{B}$ be a small category equipped with an orthogonal factorisation system $(\mathbb{E}, \mathbb{M})$. We say that $\mathbb{B}$ is **finitely copowered** if every object is the source of essentially finitely

many morphisms in $\mathbb{E}$. Furthermore, for any $\mathbb{O} \in \{\mathbb{M}, \mathbb{B}\}$, we say that $\mathbb{B}$ has *finite* $(\mathbb{O}, \mathbb{M})$ -pushout complements when any sequence $a \xrightarrow{o} b \xrightarrow{m} c$ with $o \in \mathbb{O}$ and $m \in \mathbb{M}$ has essentially finitely many pushout complements. Finally, if, additionally, $\mathbb{B}$ is $\mathbb{M}$ -adhesive, then we say that $\mathbb{B}$ is $(\mathbb{E}, \mathbb{M}, \mathbb{O})$ -well-behaved.

► **Remark 43.** When $\mathbb{O} = \mathbb{M}$, the definition is redundant since the relevant pushout complements are essentially unique [5, Theorem 4.26(4)].

► **Theorem 44.** For any $(\mathbb{E}, \mathbb{M}, \mathbb{O})$ -well-behaved category $\mathbb{B}$, both $\mathbb{M}$ and $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ have finite weak multipushouts.

This buys us, by Theorem 23,

► **Corollary 45.** In the setting of Theorem 44, for any fixed object $Z \in \mathbb{B}$, the hyperdoctrine $\mathcal{P}_Z$ on $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ has a *finitely concurrency-friendly transformer* $\text{act}_{\mathcal{P}_Z}$.

4 Syntactic conditions

In this section, we elucidate the discrepancy between *semantic* and syntactic conditional DPO rewriting, by linking the *subhom* hyperdoctrine $\mathcal{P}_Z$ with the usual presentation of conditions.

We fix a regular cardinal λ for the whole section.

4.1 λ -hyperdoctrines

Let us introduce the relaxation of hyperdoctrines announced in §2.2.

► **Definition 46.** A set is *λ -small* when its cardinality is strictly less than λ . A family is *λ -small* when it is indexed by a λ -small set. A category is *λ -small* when it has a λ -small set of objects and a λ -small set of morphisms. A functor is *λ -(co)continuous* when it preserves λ -small (co)limits, and *λ -bicontinuous* when it preserves both.

► **Definition 47.** A boolean algebra is *λ -bicomplete* when it has sups and infs of *λ -small* families. Let $\lambda\text{-CBA} \mathbf{Alg}$ denote the category of λ -bicomplete boolean algebras and λ -bicontinuous maps between them. A *λ -hyperdoctrine* is a functor $P: \mathbb{C} \rightarrow \lambda\text{-CBA} \mathbf{Alg}$, such that, for all $f \in \mathbb{C}(a, b)$, $P(f)$ has both a left and a right adjoint.

► **Remark 48.** Unlike for complete boolean algebras, since the fibres may not be (co)complete, we need to explicitly require adjoints to reindexing in order to get quantifiers. Furthermore, *λ -bicontinuity* of each $P(f)$ is redundant: it is entailed by existence of adjoints. However, we do not want morphisms in $\lambda\text{-CBA} \mathbf{Alg}$ to have adjoints, in order to be able to define hyperdoctrine *morphisms* as below.

The notion of *conditionality* generalises straightforwardly to λ -hyperdoctrines:

► **Definition 49.** Let $\mathbb{C}$ be a category equipped with a λ -small *multisquare* $K(f, g)$ for each span $b \xleftarrow{f} a \xrightarrow{g} c$. A *λ -hyperdoctrine* $\mathbb{C} \rightarrow \lambda\text{-CBA} \mathbf{Alg}$ is *K -conditional* when it satisfies the *generalised Beck-Chevalley* condition for each $K(f, g)$.

The obvious generalisation of Proposition 31 holds, which justifies dropping the K for weak multipushouts:

► **Definition 50.** A *λ -small choice of weak multipushouts* for a category $\mathbb{C}$ consists of a choice of a weak multipushout $K(f, g)$ for each span (f, g) such that $K(f, g)$ is a λ -small set.

► **Definition 51.** Given a category $\mathbb{C}$ with λ -small weak multipushouts, we call a hyperdoctrine on $\mathbb{C}$ *conditional* if it is *K-conditional* for some choice K of λ -small weak multipushouts in $\mathbb{C}$.

In particular, we obviously have:

► **Proposition 52.** For any locally small category $\mathbb{C}$ with λ -small weak multipushouts, the *subhom* λ -hyperdoctrine $\mathcal{P}_Z$ is *conditional* in the λ -hyperdoctrine sense, for any object Z of $\mathbb{C}$.

In order to apply this to categories of the form $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$, we need them to admit λ -small choices of weak multipushouts, but Theorem 44 even provides finite ones, so we readily get:

► **Proposition 53.** For any $(\mathbb{E}, \mathbb{M}, \mathbb{O})$ -well-behaved category $\mathbb{B}$, both $\mathbb{M}$ and $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ have λ -small weak multipushouts.

The next step is to organise λ -hyperdoctrines into a category:

► **Definition 54.** A *morphism* $\alpha: P \rightarrow Q$ between λ -hyperdoctrines over the same category $\mathbb{C}$ is a family of morphisms $\alpha_a: P(a) \rightarrow Q(a)$ in $\lambda\text{-CBA}lg$, such that for any $f \in \mathbb{C}(a, b)$ we have $\exists_f \circ \alpha_b = \alpha_a \circ \exists_f$.

For any λ -small choice K *multisquares* for $\mathbb{C}$, let $\lambda\text{-CHDoct}_{\mathbb{C}, K}$ denote the category of *K-conditional* λ -hyperdoctrines over $\mathbb{C}$ and λ -hyperdoctrine morphisms between them.

► **Remark 55.** For any *morphism* $\alpha: P \rightarrow Q$ and $f \in \mathbb{C}(a, b)$, one can prove that also $\forall_f \circ \alpha_b = \alpha_a \circ \forall_f$ and $\alpha_b \circ P(f) = Q(f) \circ \alpha_a$ hold. In particular, α is natural.

4.2 The initial conditional hyperdoctrine

We have now defined the category $\lambda\text{-CHDoct}_{\mathbb{C}, K}$ of *K-conditional* λ -hyperdoctrines over a given category $\mathbb{C}$ with a choice K of multisquares. This enables us to introduce our presentation of conditions, and prove that they form an initial object.

We fix a small category $\mathbb{C}$ for the whole section, together with a λ -small choice of multisquares $K(f, g)$ for each span (f, g) .

► **Definition 56.** *Raw conditions* are defined inductively by the following inference rules,

$$\frac{\dots a \vdash \varphi_i \dots (i \in I)}{a \vdash \bigvee_i \varphi_i} \quad \frac{b \vdash \psi}{a \vdash \exists_f \psi} \quad \frac{\dots a \vdash \varphi_i \dots (i \in I)}{a \vdash \bigwedge_i \varphi_i} \quad \frac{b \vdash \psi}{a \vdash \forall_f \psi}$$

where f ranges over morphisms $a \rightarrow b$ and I over λ -small set.

We let $\lambda\text{-rawcond}_{\mathbb{C}}(a)$ denote the set of raw conditions φ over a , i.e., such that $a \vdash \varphi$.

► **Remark 57.** Our presentation differs slightly from Bruggink et al.'s [2, Definition 2], but this does not affect the resulting λ -hyperdoctrine.

► **Notation 4.** We often write φ for $a \vdash \varphi$, when a is clear from context. Similarly, we often omit subscripts a introduced below.

Our next goal is to introduce a preordering relation on raw conditions, which we will then reflect into a proper ordering. For this, we need a few auxiliary operations, defined by induction on syntax: truth, falsity, negation, and the shift operation (which will become reindexing).

► **Definition 58.** For any $a \in \mathbb{C}$, let $\perp_a := \bigvee \emptyset$ and $\top_a := \bigwedge \emptyset$ (in each $\lambda\text{-rawcond}_{\mathbb{C}}(a)$).

► **Definition 59.** Let the family of *negation* endomaps $\neg_a$ on $\lambda\text{-rawcond}_{\mathbb{C}}(a)$ be defined inductively by:

$$\neg(\bigvee_i \varphi_i) = \bigwedge_i \neg\varphi_i \quad \neg\exists_f \varphi = \forall_f \neg\varphi \quad \neg(\bigwedge_i \varphi_i) = \bigvee_i \neg\varphi_i \quad \neg\forall_f \varphi = \exists_f \neg\varphi.$$

► **Definition 60.** We define the family of *shift* maps $\downarrow_g : \lambda\text{-rawcond}_{\mathbb{C}}(a) \rightarrow \lambda\text{-rawcond}_{\mathbb{C}}(c)$, for all $g : a \rightarrow c$, by induction, by

$$\begin{aligned} \downarrow_g(\bigvee_i \varphi_i) &= \bigvee_i \downarrow_g(\varphi_i), & \downarrow_g(\exists_f \psi) &= \bigvee_{(u,v) \in K(f,g)} \exists_v \downarrow_u(\psi), \\ \downarrow_g(\bigwedge_i \varphi_i) &= \bigwedge_i \downarrow_g(\varphi_i), & \downarrow_g(\forall_f \psi) &= \bigwedge_{(u,v) \in K(f,g)} \forall_v \downarrow_u(\psi). \end{aligned}$$

► **Remark 61.** For readability, the notation $\downarrow$ omits the dependency on the fixed choice $K(f, g)$ of multisquares. This dependency will disappear in our main use case anyway when we mod out by equivalence in Definition 64 below, see Proposition 68.

We may now move on to introducing our preordering relation on conditions.

► **Definition 62.** We inductively define a family of binary relations $\leq_a$ on each $\lambda\text{-rawcond}_{\mathbb{C}}(a)$ by the following inference rules (omitting subscripts in $\leq_a$ for readability),

$$\begin{array}{c} \frac{a \vdash \varphi \leq \chi \quad a \vdash \chi \leq \psi}{a \vdash \varphi \leq \psi} \quad \frac{}{a \vdash \varphi \leq \varphi} \quad \frac{\dots \quad a \vdash \varphi_i \leq \psi \quad \dots \quad (\text{for all } i)}{a \vdash \bigvee_i \varphi_i \leq \psi} \\[10pt] \frac{b \vdash \varphi \leq \downarrow_f(\psi)}{a \vdash \exists_f(\varphi) \leq \psi} \quad \frac{\dots \quad a \vdash \varphi \leq \psi_i \quad \dots \quad (\text{for all } i)}{a \vdash \varphi \leq \bigwedge_i \psi_i} \quad \frac{b \vdash \downarrow_f(\varphi) \leq \psi}{a \vdash \varphi \leq \forall_f(\psi)} \\[10pt] \frac{a \vdash \varphi \wedge \psi \leq \theta}{a \vdash \varphi \leq \neg\psi \vee \theta} \quad \frac{}{c \vdash \downarrow_g \downarrow_f \varphi \cong \downarrow_{g \circ f} \varphi} \quad \frac{}{a \vdash \downarrow_{id} \varphi \cong \varphi} \end{array}$$

where $f \in \mathbb{C}(a, b)$, $g \in \mathbb{C}(b, c)$, the symbol $\cong$ means that both $\leq$ and $\geq$ hold, and double inference rules denote the rule itself, plus its reversed version. In cases with multiple premises, this means that the conclusion entails each premise. Finally, conditions are implicitly typed; e.g., $b \vdash \varphi \leq \downarrow_f \psi$ implies $\varphi \in \lambda\text{-rawcond}_{\mathbb{C}}(b)$ and $\psi \in \lambda\text{-rawcond}_{\mathbb{C}}(a)$.

► **Remark 63.** The first two rules make each $\leq$ into a preorder. The next four define conjunction, disjunction, and quantification by the expected adjunction properties. The next one is a standard, efficient way to characterise complements. The next two rules ensure functoriality of *shift*, which entails functoriality of both quantifiers.

► **Definition 64.** For any $a \in \mathbb{C}$, let $\lambda\text{-cond}_{\mathbb{C}, K}(a)$ be the poset reflection of $\lambda\text{-rawcond}_{\mathbb{C}}(a)$:

- elements are $\cong$ -equivalence classes of *raw conditions*, which we call *conditions*;
- we have $[\varphi] \leq [\psi]$ as soon as $\varphi \leq \psi$ (for any representatives).

► **Proposition 65.** The family $\lambda\text{-cond}_{\mathbb{C}, K}$ of Definition 64, and reindexing given by *shift* as in Definition 60, forms a *K-conditional λ -hyperdoctrine*.

► **Remark 66.** The statement makes sense because shift is compatible with $\cong$, see §D.

Finally, we have:

► **Theorem 67 (Initiality).** For any small category $\mathbb{C}$ equipped with a λ -small choice K of multisquares, the *λ -hyperdoctrine* $\lambda\text{-cond}_{\mathbb{C}, K}$ is initial in $\lambda\text{-CHDoct}_{\mathbb{C}, K}$.

Let us end this subsection by showing that in the case of weak multipushouts the definition is independent of the choice, which justifies the next definition.

► **Proposition 68.** *If K and L are choices of λ -small weak multipushouts, then we have $\lambda\text{-CHDoct}_{\mathbb{C},K} = \lambda\text{-CHDoct}_{\mathbb{C},L}$ and $\lambda\text{-cond}_{\mathbb{C},K} = \lambda\text{-cond}_{\mathbb{C},L}$.*

► **Definition 69.** *For any small category $\mathbb{C}$, let $\lambda\text{-CHDoct}_{\mathbb{C}} = \lambda\text{-CHDoct}_{\mathbb{C},K}$ and $\lambda\text{-cond}_{\mathbb{C}} = \lambda\text{-cond}_{\mathbb{C},K}$, for any choice K of weak multipushouts.*

4.3 Semantics of conditions

We have now introduced the syntax and deduction rules of conditions, in a form that is very close to Bruggink et al. [2]. It remains to link this with predicates in the [subhom hyperdoctrine](#) $\mathcal{P}_Z$, i.e., sets of morphisms.

Fortunately, we have seen in Proposition 52 that each hyperdoctrine of the form $\mathcal{P}_Z$ on a category with λ -small weak multipushouts is [conditional](#). Initiality thus immediately gives a unique [conditional](#) λ -hyperdoctrine morphism $\llbracket - \rrbracket_{\mathbb{C},Z} : \lambda\text{-cond}_{\mathbb{C}} \rightarrow \mathcal{P}_Z$, which in fact gives precisely the semantics of conditions from Bruggink et al. [2, Definition 3]:

► **Proposition 70.** *The morphism $\llbracket - \rrbracket_{\mathbb{C},Z} : \lambda\text{-cond}_{\mathbb{C}} \rightarrow \mathcal{P}_Z$ satisfies*

$$\begin{aligned} k \in \llbracket \bigwedge_i \varphi_i \rrbracket &\iff \forall i, k \in \llbracket \varphi_i \rrbracket & k \in \llbracket \forall_f \varphi \rrbracket &\iff \forall g \in \mathbb{C}(b, Z), gf = k \implies g \in \llbracket \varphi \rrbracket \\ k \in \llbracket \bigvee_i \varphi_i \rrbracket &\iff \exists i, k \in \llbracket \varphi_i \rrbracket & k \in \llbracket \exists_f \varphi \rrbracket &\iff \exists g \in \mathbb{C}(b, Z), gf = k \wedge g \in \llbracket \varphi \rrbracket, \end{aligned}$$

where $f \in \mathbb{C}(a, b)$ and we omit the subscript in $\llbracket - \rrbracket_{\mathbb{C},Z}$ for readability.

4.4 Concurrency friendliness for syntactic conditions

We end this section by proving concurrency friendliness of the suitable [transformer](#) (2). This will follow by taking $P = \mathcal{P}_\emptyset$ and K some choice of weak multipushouts in:

► **Proposition 71.** *Given any small category $\mathbb{C}$ equipped with a choice K of λ -small (resp. finite) multisquares, and any K -[conditional](#) λ -hyperdoctrine $P : \mathbb{C} \rightarrow \lambda\text{-CBA}lg$, the composite*

$$\text{Cospan}(\lambda\text{-cond}_{\mathbb{C},K}) \xrightarrow{\text{Cospan}(!)} \text{Cospan}(P) \xrightarrow{\text{act}_P} \text{Sup}$$

is (resp. finitely) concurrency friendly, where the graph morphism $\text{Cospan}(!)$ maps any cospan with condition $\varphi \in \lambda\text{-cond}_{\mathbb{C},K}(m)$ to the same cospan equipped with $!(\varphi) \in P(m)$.

► **Remark 72.** When K is a choice of weak multipushouts and $P = \mathcal{P}_\emptyset$, we get the composite [transformer](#) (2) announced in §1.

5 Arrow-based vs cospan-based conditions

In this section, we exploit our framework to relate [arrow-based](#) and cospan-based conditions.

Let us fix an $(\mathbb{E}, \mathbb{M}, \mathbb{O})$ -[well-behaved](#) $\mathbb{B}$ with $\mathbb{M}$ -[initial](#) object $\emptyset$, for the whole section. We moreover let $\mathbb{C} = \mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ and let $E^{\mathbb{M}, \mathbb{O}} : \mathbb{M} \hookrightarrow \mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ denote the embedding.

Let us readily prove functoriality of the [\$\lambda\$ -hyperdoctrine](#) construction:

► **Proposition 73.** *Let $\mathbb{B}$ and $\mathbb{C}$ be categories with λ -small weak multipushouts, and let $F : \mathbb{B} \rightarrow \mathbb{C}$ be any functor preserving weak multipushouts. Then, precomposition by F extends to a functor $\Delta_F : \lambda\text{-CHDoct}_{\mathbb{C}} \rightarrow \lambda\text{-CHDoct}_{\mathbb{B}}$.*

By Propositions 52 and 53, for any distinguished object Z , we have $\mathcal{P}_Z \in \lambda\text{-CHDoct}_{\mathbb{C}}$, so by initiality we get a morphism $! : \lambda\text{-cond}_{\mathbb{C}} \rightarrow \mathcal{P}_Z$, hence $\Delta_! : \Delta_{E^{\mathbb{M},0}}(\lambda\text{-cond}_{\mathbb{C}}) \rightarrow \Delta_{E^{\mathbb{M},0}}(\mathcal{P}_Z)$.

Let us now introduce the morphism $\mathcal{P}\alpha^{\mathbb{M},0}$ announced in §2.3:

► **Definition 74.** We define a natural transformation

$$\begin{array}{ccc} & \text{Cospan}(\mathbb{M}, \mathbb{O}) & \\ E^{\mathbb{M},0} \nearrow & \Downarrow \alpha^{\mathbb{M},0} & \searrow y_0 \\ \mathbb{M} & \xrightarrow{\coprod_{b' \in \mathbb{B}} y_{b'}} & \text{Set}^{op} \end{array} \quad \text{by} \quad \alpha_b^{\mathbb{M},0} : \coprod_{b'} \mathbb{M}(b, b') \rightarrow \text{Cospan}(\mathbb{M}, \mathbb{O})(b, \emptyset)$$

$$(b', m) \mapsto [b \xrightarrow{m} b' \xleftarrow{!} \emptyset]$$

We obtain the following, which makes sense by Proposition 53 and the fact that $\Delta_{E^{\mathbb{M},0}}\mathcal{P}_0$ is just the composite $\mathbb{M} \xrightarrow{E^{\mathbb{M},0}} \text{Cospan}(\mathbb{M}, \mathbb{O}) \xrightarrow{y_0} \text{Set}^{op} \xrightarrow{\mathcal{P}} \lambda\text{-CBA}lg$.

► **Proposition 75.** The embedding $E^{\mathbb{M},0}$ preserves weak multipushouts, and the natural transformation $\mathcal{P}\alpha^{\mathbb{M},0} : \Delta_{E^{\mathbb{M},0}}\mathcal{P}_0 \rightarrow \mathcal{P}_{\coprod}$ is a morphism of *conditional* λ -hyperdoctrines.

We thus conclude by initiality:

► **Corollary 76.** The diagram below left commutes, where the top-right-bottom composite may be pictured as below right.

$$\begin{array}{ccc} \lambda\text{-cond}_{\mathbb{M}} & \xrightarrow{!} & \Delta_{E^{\mathbb{M},0}}(\lambda\text{-cond}_{\mathbb{C}}) \\ \downarrow & & \downarrow \Delta_{E^{\mathbb{M},0}}(!) \\ \mathcal{P}_{\coprod} & \xleftarrow{\mathcal{P}\alpha^{\mathbb{M},0}} & \Delta_{E^{\mathbb{M},0}}\mathcal{P}_Z \end{array} \quad \begin{array}{ccc} \mathbb{M} & \xrightarrow{\coprod_{b'} y_{b'}} & \text{Set}^{op} \\ \downarrow E^{\mathbb{M},0} & \nearrow \alpha^{\mathbb{M},0} & \downarrow y_0 \\ \lambda\text{-cond}_{\mathbb{M}} & \xrightarrow{!} & \mathbb{C} \xrightarrow{y_0} \text{Set}^{op} \\ & \searrow \lambda\text{-cond}_{\mathbb{C}} & \downarrow \varphi \\ & & \lambda\text{-CBA}lg \end{array}$$

6 Conclusion

We proposed an abstract account of the concurrency theorem from conditional DPO rewriting, which works for any (complete, boolean) hyperdoctrine satisfying a *generalised Beck-Chevalley* condition. This result readily instantiates to a semantic form of the concurrency theorem, in which conditions are sets of morphisms (or rather isomorphism-classes of cospans).

We have furthermore reconstructed the syntactic version of conditions by exhibiting them as an initial *conditional* λ -hyperdoctrine. We then deduced the usual concurrency theorem featuring syntactic conditions, by a suitable change of base (Proposition 71).

Finally, we have recovered the known result that *arrow-based* conditions are subsumed by cospan-based ones, using initiality.

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A Frobenius reciprocity

In this section, we recall that any (complete) hyperdoctrine satisfies Frobenius reciprocity. Let us start by recalling the definition.

► **Definition 77.** A complete hyperdoctrine $P: \mathbb{C} \rightarrow \mathbf{CBA}lg$ satisfies *Frobenius reciprocity* iff, for all morphisms $f: c \rightarrow d$, $\varphi \in P(c)$, and $\psi \in P(d)$, the canonical inequality

$$\exists_f(f^*(\varphi) \wedge \psi) \leq \varphi \wedge \exists_f(\psi) \tag{6}$$

is actually an equality.

► **Proposition 78.** Any hyperdoctrine $P: \mathbb{C} \rightarrow \mathbf{CBA}lg$ satisfies Frobenius reciprocity.

Proof. By adjunction, our goal $\varphi \wedge \exists_f(\psi) \leq \exists_f(f^*(\varphi) \wedge \psi)$ is equivalent to $f^*(\varphi \wedge \exists_f(\psi)) \leq f^*(\varphi) \wedge \psi$. But we have

$$\begin{aligned} f^*(\varphi \wedge \exists_f(\psi)) &= f^*(\varphi) \wedge f^*(\exists_f(\psi)) && \text{(by continuity of right adjoints)} \\ &\leq f^*(\varphi) \wedge \psi && \text{(by adjunction)} \end{aligned}$$

as desired. $\blacktriangleleft$

B Proofs of §1

We work towards the proof of Proposition 18, starting with a well-known result.

► **Lemma 79.** *In any frame, i.e., distributive sup-lattice, if x is an atom, then $x \leq \bigvee_i y_i$ if and only if there is some i such that $x \leq y_i$.*

Proof. We prove the only if direction which is the hard one. Note that $p \leq q$ is equivalent to $p = p \wedge q$. Now, assume $x \leq \bigvee_i y_i$, i.e., $x \wedge (\bigvee_i y_i) = x$. By distributivity, this means that $\bigvee_i x \wedge y_i = x$. Because x is an atom, each $x \wedge y_i$ is either $\perp$ or x . But since $x \neq \perp$, at least one of these $x \wedge y_i$ must be x . In other words, there exists i such that $x \wedge y_i = x$, i.e., $x \leq y_i$. $\blacktriangleleft$

► **Proposition 18.** *For any concurrency-friendly transformer $F: G \rightarrow \mathbf{Sup}$ such that each $F(a)$ is atomic distributive (e.g., a power set), the induced labelled transition system $\tilde{F}$ is concurrency-friendly.*

Proof. We show that $\tilde{F}$ is concurrency-friendly for the same set of composite edges that makes F concurrency-friendly, as a transformer. Assume that $x \xRightarrow{\rho_1} y \xRightarrow{\rho_2} z$ for some atoms x, y , and z , i.e., $y \leq \rho_1 \cdot x$ and $z \leq \rho_2 \cdot y$. Let us show that $z \leq \rho \cdot x$ for some $\rho \in \rho_2 \circ \rho_1$. First, we have $z \leq \rho_2 \cdot \rho_1 \cdot x$. Since F is concurrency-friendly, this entails that $z \leq \bigvee_{\rho \in \rho_2 \circ \rho_1} \rho \cdot x$. By Lemma 79, there exists $\rho \in \rho_2 \circ \rho_1$ such that $z = z \wedge (\rho \cdot x)$, i.e., $z \leq \rho \cdot x$.

Conversely, assume that $x \leq \rho \cdot z$, for some $\rho \in \rho_2 \circ \rho_1$ and atoms x, y . Let us show that there exists an atom y such that $y \leq \rho_1 \cdot x$ and $z \leq \rho_2 \cdot y$. Since F is concurrency-friendly, $F(\rho) \leq F(\rho_2) \circ F(\rho_1)$ and thus $z \leq \rho_2 \cdot \rho_1 \cdot x$. Now, because F is pointwise atomic, we can decompose $\rho_1 \cdot x$ as a join of atoms $\bigvee_i y_i$. Since ρ_2 preserves joins, we get $z \leq \bigvee_i \rho_2 \cdot y_i$. By Lemma 79, there exists i such that $z \leq \rho_2 \cdot y_i$. We take y to be y_i . $\blacktriangleleft$

C Proofs for §3

► **Lemma 28.** *Let $\mathbb{C}$ denote any category equipped with a choice $K(f, g)$ of multisquare for each span $b \xleftarrow{f} a \xrightarrow{g} c$. Then, for any K -conditional hyperdoctrine P on $\mathbb{C}$, the transformer act_P is concurrency-friendly. The composite of any two P -rules given by*

$$\left(a \xrightarrow{l} m \xleftarrow{r} b, \quad \varphi \in P(m) \right) \quad \text{and} \quad \left(b \xrightarrow{l'} n \xleftarrow{r'} c, \quad \psi \in P(n) \right),$$

is the set of rules of the form

$$\left(a \xrightarrow{l} m \xrightarrow{u} p \xleftarrow{v} n \xleftarrow{r'} c, \quad u^*(\varphi) \wedge v^*(\psi) \in P(p) \right),$$

where $(u, v) \in K(r, l')$. In particular, if each $K(f, g)$ is finite, then act_P is finitely concurrency-friendly.

Proof. The claim follows via the [generalised Beck-Chevalley](#) condition (step (★)), [Frobenius reciprocity](#) (step (★★)), and finally functoriality of $\exists$ and $-^*$ as well as distributivity of $-^*$ (step (★★★)):

$$\begin{aligned}
 \text{act}_P(\rho_2)\text{act}_P(\rho_1) &= \exists_{r'}(\psi \wedge l'^*(\exists_r(\varphi \wedge l^*(.)))) \\
 &\stackrel{(\star)}{=} \bigvee_{(u,v) \in K(r,l')} \exists_{r'}(\psi \wedge \exists_v u^*(\varphi \wedge l^*(.))) \\
 &\stackrel{(\star\star)}{=} \bigvee_{(u,v) \in K(r,l')} \exists_{r'} \exists_v (v^* \psi \wedge (u^* \varphi \wedge u^* l^*(.))) \\
 &\stackrel{(\star\star\star)}{=} \bigvee_{(u,v) \in K(r,l')} \exists_{r';v} ((v^* \psi \wedge u^* \varphi) \wedge (l; u)^*(.))
 \end{aligned}$$

◀

We prove a generalisation of Proposition 31.

► **Lemma 80.** *Let $P: \mathbb{C} \rightarrow \mathbf{CBA} \mathbf{lg}$ (resp. $P: \mathbb{C} \rightarrow \lambda\text{-CBA} \mathbf{lg}$) be a hyperdoctrine (resp. λ -hyperdoctrine) with two choices K and L of weak multipushouts (resp. λ -small weak multipushouts). Then P is K -conditional iff it is L -conditional.*

Proof. By symmetry, it is enough to show that P is K -conditional if it is L -conditional. Let us consider any span $b \xleftarrow{f} a \xrightarrow{g} c$ in $\mathbb{C}$. We need to show

$$g^* \circ \exists_f \leq \bigvee_{(u,v) \in K(f,g)} (\exists_v \circ u^*).$$

By conditionality of P , we have

$$g^* \circ \exists_f \leq \bigvee_{(u',v') \in L(f,g)} (\exists_{v'} \circ (u')^*).$$

But, by preservation of weak multipushouts, we have that, for any $(u',v') \in L(f,g)$, there exists $(u,v) \in K(f,g)$ and m making both triangles below commute,

$$\begin{array}{ccccc}
 & & x & & \\
 & u \nearrow & & \nwarrow v & \\
 b & & & & c \\
 & u' \searrow & & \swarrow v' & \\
 & & y & &
 \end{array}$$

so that we have

$$\exists_{v'} \circ (u')^* = \exists_v \exists_m m^* u^* \leq \exists_v u^*.$$

This readily entails

$$\bigvee_{(u',v') \in L(f,g)} (\exists_{v'} \circ (u')^*) \leq \bigvee_{(u,v) \in K(f,g)} (\exists_v \circ u^*)$$

hence the result. ◀

► **Lemma 34.** *For any locally small category $\mathbb{C}$, $Z \in \mathbb{C}$, $f \in \mathbb{C}(a,b)$, $\varphi \in \mathcal{P}_Z(a)$, and $\psi \in \mathcal{P}_Z(b)$, we have:*

- $f^*(\varphi)$ is the set of extensions of morphisms in φ along f ;
- $\exists_f(\psi)$ is the set of composites of f with some morphism in ψ , or equivalently the set of morphisms $a \rightarrow Z$ admitting an extension along f that is in ψ ;
- $\forall_f(\psi)$ is the set of morphisms $a \rightarrow Z$ whose extensions along f are all in ψ .

In symbols:

$$\begin{aligned}
 f^*(\varphi) &= \mathbb{C}(f, Z)^{-1}(\varphi) &= \{k \in \mathbb{C}(b, Z) \mid k \circ f \in \varphi\} \\
 \exists_f(\psi) &= \mathbb{C}(f, Z)(\psi) &= \{h \in \mathbb{C}(a, Z) \mid \exists k \in \mathbb{C}(b, Z), h = k \circ f \wedge k \in \psi\} \\
 \forall_f(\psi) & &= \{h \in \mathbb{C}(a, Z) \mid \forall k \in \mathbb{C}(b, Z), h = k \circ f \implies k \in \psi\}.
 \end{aligned}$$

Proof sketch. The case of f^* is by definition; both other cases follow easily by adjunction. ◀

► **Proposition 33.** For any locally small category $\mathbb{C}$ with weak multipushouts and distinguished object Z , the *subhom* hyperdoctrine $\mathcal{P}_Z$ is *conditional*.

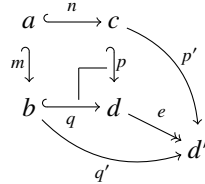
Proof sketch. In the situation of Definition 27, for any $\varphi \in \mathcal{P}_Z(c)$,

- $g^*(\exists_f(\varphi))$ is the set of morphisms $v' \in \mathbb{C}(c, Z)$ for which there exists $u' \in \varphi$ making the square commute, while
- $\bigvee_{(u,v) \in K(f,g)} \exists_v(u^*(\varphi))$ consists of all $v' \in \mathbb{C}(c, Z)$ admitting an extension w along v such that $w \circ u \in \varphi$.

The two conditions are straightforwardly equivalent by definition of weak multipushouts. ◀

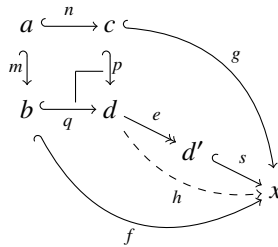
Working towards the proof of Theorem 44, we have:

► **Definition 81.** In the setting of Theorem 44, for any span (m, n) as below, let $\kappa^{\mathbb{E}, \mathbb{M}}(m, n)$ denote the finite set obtained by choosing a pushout d as shown, post-composing with a chosen representative of each isomorphism-class of $e \in d/\mathbb{E}$ as below, and keeping only those squares for which both p' and q' are in $\mathbb{M}$.



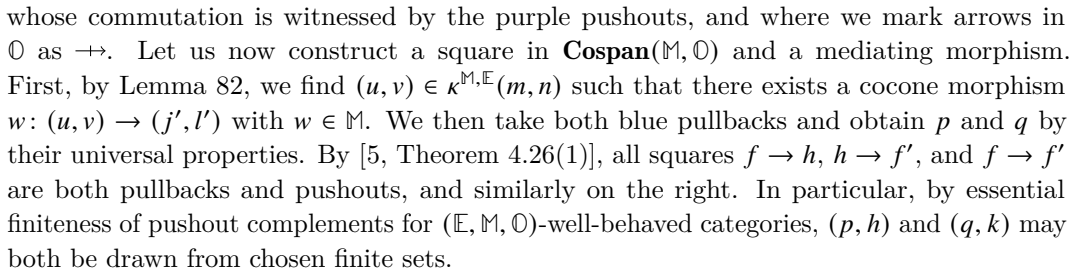
► **Lemma 82.** Each multisquare $\kappa^{\mathbb{E}, \mathbb{M}}(m, n)$ forms a weak multipushout of (m, n) in $\mathbb{M}$.

Proof. Indeed, for any cocone as below with $f, g \in \mathbb{M}$,



we find a mediating morphism h as shown by universal property of pushout. Factoring h as $s \circ e$, we get by *left cancellation* that ep and eq are in $\mathbb{M}$, so that we have found a factorisation of (f, g) through $(ep, eq) \in \kappa^{\mathbb{M}, \mathbb{E}}(m, n)$, via s . ◀

Let us now consider a commuting square in $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$, as the exterior of the diagram below,



It remains to show that commuting squares of the form of the top face above are essentially finitely many. But $\kappa^{\mathbb{M}, \mathbb{E}}(m, n)$ is finite, so there are finitely many pairs (u, v) . Furthermore, we saw that pairs (p, h) and (q, k) may be drawn from chosen finite sets. Finally, the pullback is essentially unique, which concludes the proof. $\blacktriangleleft$

► **Remark 83.** For the case $\mathbb{O} = \mathbb{B}$, i.e., for “non-linear” cospans, we are currently unaware of a general classification of finitary $\mathbb{M}$ -adhesive categories that are $(\mathbb{E}, \mathbb{M}, \mathbb{B})$ -well behaved. However, there are known examples from [1, Sec. 5.3.3]. In these examples, the base category $\mathbb{B}$ has an $\mathbb{M}$ -partial map classifier and hence has final pullback complements along $\mathbb{M}$ -morphisms. It is then demonstrated in [1, Prop. 5.26] that all pushouts under a sequence (f, m) of a generic morphism f followed by an $\mathbb{M}$ -morphism m arise as an $(\mathbb{M}, \mathbb{B})$ -factorization $n = n'; n''$ of the $\mathbb{M}$ -morphism n that arises from taking a final pullback (n, g) of (f, m) . For the specific examples of the category of directed multigraphs (which is a category of covariant preheaves, and which is adhesive [13]) and for the category of directed simple graphs (which is a quasitopos, and which is thus $\mathbb{M}$ -adhesive for $\mathbb{M}$ the class of regular monos [10, Lem. 13], [7, Thms. 3.4 &

3.6]), passing to their respective finitary restrictions entails that the FPC-based construction yields essentially finite sets of pushouts (f, m) with f in $\mathbb{B}$ and m in $\mathbb{M}$ (cf. [1, Sec. 5.3.3] for some concrete illustrations). Many $\mathbb{M}$ -adhesive categories of interest in rewriting theory are in fact quasi-topoi (cf. [1, Table 2]), hinting of further interesting examples of categories that are $(\mathbb{E}, \mathbb{M}, \mathbb{B})$ -well behaved, but a detailed technical analysis of this phenomenon is postponed to future work.

D Proofs for §4

The fact that $\lambda\text{-cond}_{\mathbb{C}, K}$ forms a (conditional) hyperdoctrine relies on the following lemma.

► **Lemma 84.** *For any $f: A \rightarrow B$, the following rule is derivable.*

$$\frac{A \vdash \varphi \leq \psi}{B \vdash \downarrow_f(\varphi) \leq \downarrow_f(\psi)}$$

Proof. We have

$$\frac{\frac{\frac{A \vdash \varphi \leq \psi}{A \vdash \psi \leq \forall_f \downarrow_f(\psi)} \text{ ADJUNCTION} \quad \frac{B \vdash \downarrow_f \psi \leq \downarrow_f(\psi)}{A \vdash \psi \leq \forall_f \downarrow_f(\psi)} \text{ REFLEXIVITY}}{A \vdash \varphi \leq \forall_f \downarrow_f(\psi)} \text{ TRANSITIVITY} \quad \frac{A \vdash \varphi \leq \forall_f \downarrow_f(\psi)}{B \vdash \downarrow_f(\varphi) \leq \downarrow_f(\psi)} \text{ ADJUNCTION}$$

► **Proposition 65.** *The family $\lambda\text{-cond}_{\mathbb{C}, K}$ of Definition 64, and reindexing given by [shift](#) as in Definition 60, forms a K -[conditional \$\lambda\$ -hyperdoctrine](#).*

Proof. The inference rules make each poset $\lambda\text{-cond}_{\mathbb{C}, K}(A)$ a λ -[bicomplete](#) boolean algebra by construction. Furthermore, Lemma 84 provides a candidate action on morphisms given by $\downarrow$. The rules ensure it is functorial and that it preserves joins and meets. We thus have a functor $\lambda\text{-cond}_{\mathbb{C}, K}: \mathbb{C} \rightarrow \lambda\text{-CBA} \mathbf{lg}$. Finally, [Beck-Chevalley](#) is satisfied for canonical multisquares by definition of [shift](#). ◀

We prove Theorem 67, recalled below.

► **Theorem 67** (Initiality). *For any small category $\mathbb{C}$ equipped with a λ -small choice K of [multisquares](#), the λ -[hyperdoctrine](#) $\lambda\text{-cond}_{\mathbb{C}, K}$ is initial in $\lambda\text{-CHDoct}_{\mathbb{C}, K}$.*

Proof. We want to prove that, given any λ -[hyperdoctrine](#) Q on $\mathbb{C}$, there is a unique morphism $\lambda\text{-cond}_{\mathbb{C}, K} \rightarrow Q$. That is, for each object A of $\mathbb{C}$, we need to show that there exists a λ -[bicontinuous](#) functor $F_A: \lambda\text{-cond}_{\mathbb{C}, K}(A) \rightarrow Q(A)$, that this assignment commutes with $\forall$ and $\exists$, and that such an assignment is unique. The key observation is that $\lambda\text{-cond}_{\mathbb{C}, K}(A)$ is the free poset on $\lambda\text{-rawcond}_{\mathbb{C}}(A)$, that is, the embedding $\lambda\text{-rawcond}_{\mathbb{C}}(A) \rightarrow \lambda\text{-cond}_{\mathbb{C}, K}(A)$ uniquely factors any functor from $\lambda\text{-rawcond}_{\mathbb{C}}(A)$, seen as a (preordered) category, to some poset X . Moreover, because this embedding is an equivalence of categories, $\lambda\text{-cond}_{\mathbb{C}, K}(A)$ is even the free λ -[bicomplete](#) poset on $\lambda\text{-rawcond}_{\mathbb{C}}(A)$: any λ -bicontinuous functor from $\lambda\text{-rawcond}_{\mathbb{C}}(A)$ to some λ -bicomplete poset X uniquely factors as $\lambda\text{-rawcond}_{\mathbb{C}}(A) \rightarrow \lambda\text{-cond}_{\mathbb{C}, K}(A)$ followed by some λ -bicontinuous functor.

Now, we define an obvious family of λ -bicontinuous functors F_A^0 from $\lambda\text{-rawcond}_{\mathbb{C}}(A)$ to $Q(A)$ as follows. First, we define a mapping on objects by recursion conditions. Conditionality of Q ensures that the shift of a condition is interpreted as the shift of its interpretation. Then, we define the action on morphisms, by recursion on the definition of the entailment

relation. The resulting family of functors $(F_A^0)_A$ commute with $\forall$ and $\exists$ by construction. Each $\lambda\text{-rawcond}_{\mathbb{C}}(A)$ and F_A^0 in fact live in the category of preordered-based λ -bicomplete Boolean algebras (defined exactly like λ -bicomplete Boolean algebras, but replacing posets with preorders) and λ -bicontinuous functors between them.

Using the above-mentioned universal property of each $\lambda\text{-cond}_{\mathbb{C},K}(A)$, we readily obtain a family $(F_A: \lambda\text{-cond}_{\mathbb{C},K}(A) \rightarrow \mathcal{Q}(A))_A$ of λ -bicontinuous functors. Moreover, by universal property of $\lambda\text{-cond}_{\mathbb{C},K}(A)$ as a free poset over $\lambda\text{-rawcond}_{\mathbb{C}}(A)$, F commutes with $\forall$ and $\exists$. Both commutations ensure that F commutes with shift as well: the two required inequalities are the mate inequalities of commutation with $\forall$ and $\exists$. This proves existence of the desired morphism.

Uniqueness follows again by universal property of each $\lambda\text{-cond}_{\mathbb{C},K}(A)$. $\blacktriangleleft$

► **Proposition 68.** *If K and L are choices of λ -small weak multipushouts, then we have $\lambda\text{-CHDoct}_{\mathbb{C},K} = \lambda\text{-CHDoct}_{\mathbb{C},L}$ and $\lambda\text{-cond}_{\mathbb{C},K} = \lambda\text{-cond}_{\mathbb{C},L}$.*

Proof. By Lemma 80, objects of $\lambda\text{-CHDoct}_{\mathbb{C},L}$ are the same as those of $\lambda\text{-CHDoct}_{\mathbb{C},K}$. Since the morphisms are also the same, the initiality properties of $\lambda\text{-cond}_{\mathbb{C},K}$ and $\lambda\text{-cond}_{\mathbb{C},L}$ are actually the same and so they are isomorphic. But because they are both quotients of the same family of sets $(\lambda\text{-rawcond}_{\mathbb{C}}(c))_c$, they are equal. $\blacktriangleleft$

► **Proposition 70.** *The morphism $\llbracket - \rrbracket_{\mathbb{C},Z}: \lambda\text{-cond}_{\mathbb{C}} \rightarrow \mathcal{P}_Z$ satisfies*

$$\begin{aligned} k \in \llbracket \bigwedge_i \varphi_i \rrbracket &\iff \forall i, k \in \llbracket \varphi_i \rrbracket & k \in \llbracket \forall_f \varphi \rrbracket &\iff \forall g \in \mathbb{C}(b, Z), gf = k \implies g \in \llbracket \varphi \rrbracket \\ k \in \llbracket \bigvee_i \varphi_i \rrbracket &\iff \exists i, k \in \llbracket \varphi_i \rrbracket & k \in \llbracket \exists_f \varphi \rrbracket &\iff \exists g \in \mathbb{C}(b, Z), gf = k \wedge g \in \llbracket \varphi \rrbracket, \end{aligned}$$

where $f \in \mathbb{C}(a, b)$ and we omit the subscript in $\llbracket - \rrbracket_{\mathbb{C},Z}$ for readability.

Proof. As a morphism, $\llbracket - \rrbracket_{\mathbb{C},Z}$ satisfies the following,

$$\llbracket \bigwedge_i \varphi_i \rrbracket = \bigcap_i \llbracket \varphi_i \rrbracket \quad \llbracket \forall_f \varphi \rrbracket = \forall_f \llbracket \varphi \rrbracket \quad \llbracket \bigvee_i \varphi_i \rrbracket = \bigcup_i \llbracket \varphi_i \rrbracket \quad \llbracket \exists_f \varphi \rrbracket = \exists_f \llbracket \varphi \rrbracket.$$

The result then follows from Lemma 34. $\blacktriangleleft$

► **Proposition 71.** *Given any small category $\mathbb{C}$ equipped with a choice K of λ -small (resp. finite) multisquares, and any K -conditional λ -hyperdoctrine $P: \mathbb{C} \rightarrow \lambda\text{-CBA} \mathbf{Alg}$, the composite*

$$\text{Cospan}(\lambda\text{-cond}_{\mathbb{C},K}) \xrightarrow{\text{Cospan}(!)} \text{Cospan}(P) \xrightarrow{\text{act}_P} \mathbf{Sup}$$

is (resp. finitely) concurrency friendly, where the graph morphism $\text{Cospan}(!)$ maps any cospan with condition $\varphi \in \lambda\text{-cond}_{\mathbb{C},K}(m)$ to the same cospan equipped with $!(\varphi) \in P(m)$.

Proof. Consider two composable edges ρ_1 and ρ_2 in $\text{Cospan}(\lambda\text{-cond})$. We define $\rho_2 \circ \rho_1$ by the same formula as in the proof of Theorem 19. It is straightforward to check that its image by $\text{Cospan}(!)$ is $\text{Cospan}(!)(\rho_2) \circ \text{Cospan}(!)(\rho_1)$. The result follows from P being finitely concurrently-friendly. $\blacktriangleleft$

E Proofs for §5

► **Definition 85.** *We denote by $\lambda\text{-HDoct}_{\mathbb{C}}$ the category of λ -hyperdoctrines on $\mathbb{C}$ and morphisms between them.*

► **Proposition 73.** *Let $\mathbb{B}$ and $\mathbb{C}$ be categories with λ -small weak multipushouts, and let $F: \mathbb{B} \rightarrow \mathbb{C}$ be any functor preserving weak multipushouts. Then, precomposition by F extends to a functor $\Delta_F: \lambda\text{-CHDoct}_{\mathbb{C}} \rightarrow \lambda\text{-CHDoct}_{\mathbb{B}}$.*

Proof. First of all, F clearly induces a functor $\Delta_F: \lambda\text{-HDoct}_{\mathbb{C}} \rightarrow \lambda\text{-HDoct}_{\mathbb{B}}$ between categories of plain λ -hyperdoctrines. The statement thus reduces to proving that each $\Delta_F(P)$ is conditional. Let us consider any span $b \xleftarrow{f} a \xrightarrow{g} c$ in $\mathbb{B}$. We need to show

$$F(g)^* \circ \exists_{F(f)} \leq \bigvee_{(u,v) \in K(f,g)} (\exists_{F(v)} \circ F(u)^*).$$

This follows from P being $F(K)$ -conditional by Lemma 80. ◀

► **Proposition 24.** *The λ -hyperdoctrine $\mathcal{P}_{\sqcup}$ on $\mathbb{M}$ is *conditional*.*

Proof. It is easy to see that *conditional* λ -hyperdoctrines are stable under products. The result follows from Proposition 52 and $\mathcal{P}_{\sqcup}$ being isomorphic to the product $\prod_{b \in \mathbb{B}} \mathcal{P}_b$. ◀

► **Proposition 75.** *The embedding $E^{\mathbb{M},0}$ preserves weak multipushouts, and the natural transformation $\mathcal{P}\alpha^{\mathbb{M},0}: \Delta_{E^{\mathbb{M},0}}\mathcal{P}_0 \rightarrow \mathcal{P}_{\sqcup}$ is a morphism of *conditional* λ -hyperdoctrines.*

Proof. First we show that $E^{\mathbb{M},0}: \mathbb{M} \hookrightarrow \mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ preserves weak multipushouts. We rely on the fact that $\mathbf{Cospan}(\mathbb{M}, \mathbb{O})$ has an orthogonal factorisation system: each cospan $a \xrightarrow{f} b \xleftarrow{g} c$ is the composition of $f^* = (a \xrightarrow{f} b \xleftarrow{=} b)$ and $g_* = (b \xrightarrow{=} b \xleftarrow{g} c)$. This decomposition is unique up to unique isomorphisms, that is, if there exists $f^*; g_* = f'^*; g'_*$, with some $f' \in \mathbb{M}(a, b')$ and $g' \in \mathbb{O}(c, b')$, then there exists a unique isomorphism $i: b \cong b'$ such that $f' = f; i$ and $g' = g; i$. Note that f^* is precisely $E^{\mathbb{M},0}f$. Now, consider a weak multipushout K of a span $b \xleftarrow{f} a \xrightarrow{g} c$ in $\mathbb{M}$. Let us show that its image by $E^{\mathbb{M},0}$ is also a weak multipushout. Consider a cocone, that is, cospan $(b \xrightarrow{m^*} x \xrightarrow{m'_*} e \xleftarrow{n'_*} y \xleftarrow{n^*} c)$ such that $f^*; m^*; m'_* = g^*; n^*; n'_*$, or equivalently, $(f; m)^*; m'_* = (g; n)^*; n'_*$. We need to find some $(u, v) \in K$ such that (u^*, v^*) factors this cocone.

By uniqueness of the decomposition, we have an isomorphism $i: x \cong y$ such that $f; m; i = g; n$ and $m'; i = n'$. Therefore, there exists $(b \xrightarrow{u} d \xleftarrow{v} c) \in K$ and $r: d \rightarrow y$ such that $m; i = u; r$ and $n = v; r$. We now show that (u^*, v^*) factors the cocone, with $r^*; n'_*$ as mediating arrow.

$$\begin{aligned} m^*; m'_* &= m^*; i_*; (i^{-1})^*; m'_* \\ &= u^*; r^*; i_*; m'_* \\ &= u^*; r^*; (m'; i)_* \\ &= u^*; r^*; n'_* \\ n^*; n'_* &= v^*; r^*; n'_* \end{aligned}$$

It remains to show that $\mathcal{P}\alpha^{\mathbb{M},0}: \Delta_{E^{\mathbb{M},0}}\mathcal{P}_0 \rightarrow \mathcal{P}_{\sqcup}$ is a morphism of *conditional* λ -hyperdoctrines. The result follows from the fact that each naturality square of $\alpha^{\mathbb{M},0}$ is a weak pullback in the category of sets. Indeed, by applying Proposition 33 with $\mathbb{C} = \mathbf{Set}^{op}$, $Z = 1$, and K any global choice of weak pushouts in $\mathbf{Set}^{op}$, we obtain that weak pullbacks of sets are such that the mates of their images by $\mathcal{P}: \mathbf{Set}^{op} \rightarrow \mathbf{CAlg}$ are identities (in fact this exactly characterises weak pullbacks of sets). ◀