Renormalization and Redundancy

based on **1310.4185**

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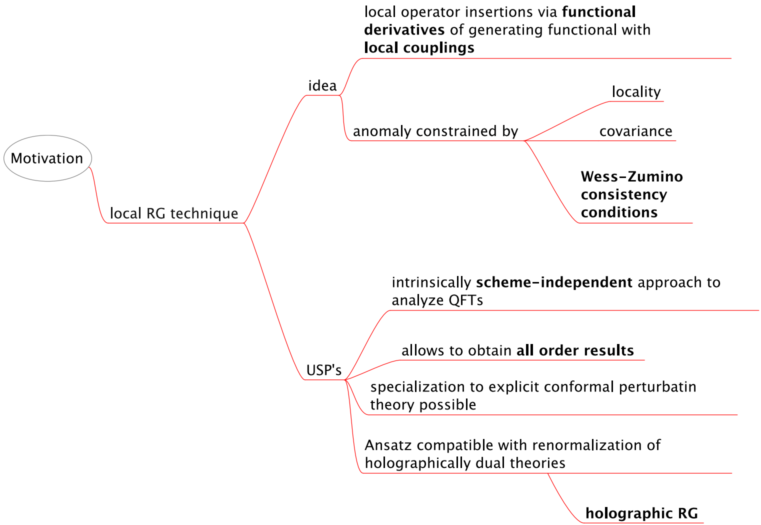
Part I: Osborn's local renormalization group

Motivation for Osborn's local renormalization group



General concept

- **Philosophy**: define QFTs through correlation functions obtianed via functional differentiation of some generating functional defined in terms of **local sources**
- as compared to the ordinary Wilsonian approach, e.g. for a classically Weyl-invariant theory (invariant under $g_{\mu\nu} \to e^{-2\sigma(x)} g_{\mu\nu}$), for which classically

$$T_\mu^\mu \equiv \Theta = 0$$

  one must carefully implement a **regularization** and **renormalization scheme** to obtain the QFT expressions, e,g,

$$\Theta = \beta^i \Phi_i + \text{local terms}$$

  (reason for local terms: β-functions only capture the effect of **constant** rescalings of the metric)
- **Problem**: intrinsically perturbative method, in general only one-loop order reasonably tractable

Osborn 1991: the local RG equation

- **key idea**: instead of implementing counter-terms explicitly, introduce **local couplings** $\lambda^i(x)$ and assume the existence of some **generating functional** W **of connected correlation functions**, from which arbitrary insertions of **renormalized** local operators $\Theta(x)$ and $\Phi_i(x)$ may be obtained via **functional differentiation** w.r.t. the local sources (**Schwinger's action principle**):

$$\langle[\Phi_i(x)]\rangle = \frac{\delta}{\delta\lambda^i(x)} W \,, \quad \langle[\Theta(x)]\rangle = \mu(x)\frac{\delta}{\delta\mu(x)} W$$

with $W \equiv W[\lambda^i(x), \mu(x)]$ defined via

$$e^W = \int D[\phi] e^{-\frac{1}{\ell} S_0}$$

- here, the QFT action S_0 contains all dimension $\leq d$ counterterms necessary to ensure that correlators are properly defined distributions and that W remains a **finite funtional to all orders in perturbation theory**

Osborn 1991: the local RG equation

- **Schwinger's action principle**:

$$\langle[\Phi_i(x)]\rangle = \frac{\delta}{\delta\lambda^i(x)} W \,, \quad \langle[\Theta(x)]\rangle = \mu(x)\frac{\delta}{\delta\mu(x)} W$$

  with $e^W = \int D[\phi] e^{-\frac{1}{\ell} S_0}$

- the counterterms necessary to define insertions of local operators are absorbed implicitly in the definition of $\phi_i \equiv [\Phi_i]$ etc. …
- … but there must be **additional local counterterms** included in the definition of the QFT to account for the curved space and for the local couplings λ^i, aka **contact terms**
- Osborn's method leads to **all-order generalizations of the statements about the one-loop quantum anaomalies**

The local Callan-Symanzik equation

Effect of a local scale transformation $g_{\mu\nu} \to e^{-\sigma(x)} g_{\mu\nu}$ in a **2d** QFT:

$$(\Delta_\sigma^W - \Delta_\sigma^\beta) W = -\frac{1}{\ell} \int dv\, \sigma \left(\frac{1}{2} \beta^\Phi R - \frac{1}{2} \chi_{ij} \partial_\mu \lambda^i \partial^\mu \lambda^j \right) + \frac{1}{\ell} \int dv\, \partial_\mu \sigma\, w_i \partial^\mu \lambda^i$$

- with:
  - $\Delta_\sigma^W := 2 \int dv \sigma g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)}$ ($v := d^2 x \sqrt{g}$)
  - $\Delta_\sigma^\beta := \int dv \sigma \beta^i \frac{\delta}{\delta \lambda^i(x)}$
  - $R \equiv \mu^2 R_2(x) = -2 \partial_{mu} \partial^\mu \ln \mu(x)$ (2d curvature density)
- amounts to **integrated quantum anomaly**
- **all-order generalization** of the one-loop 2d trace anomaly (i.e. of the term $\frac{1}{2} \beta^\Phi R$ for constant rescalings $\sigma = const$)
- possible terms in the anomaly constrained by
  - **locality**
  - **2d covariance**
  - **Wess-Zumino conditions**: $[\Delta_\sigma^W - \Delta_\sigma^\beta, \Delta_{\sigma'}^W - \Delta_{\sigma'}^\beta] W \stackrel{!}{=} 0$

Most successful results for $2d$ QFTs: RG flow equations

- general setup: 2d Euclidean CFT S_0, perturbed by scalars ϕ_i:

$$\delta S = \int d^2x \, \lambda^i \phi_i(x)$$

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- **RG flow equations**:
    - Osborn 1991: alternative proof of **c-theorem** (Zamolodchikov 1986):

    $$\mu \frac{\partial c}{\partial \mu} = -\beta^i \mathbf{g}_{ij} \beta^j$$

    μ – RG scale, c – c-function, β^i – components of β-function vector field, g_{ij} – Zamolodchikov–metric

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  - **gradient formula** (Friedan & Konechny 2009)

$$\frac{\partial c}{\partial \lambda^i} = -(g_{ij} + \Delta g_{ij})\beta^j - b_{ij}\beta^j$$

    with: Δg_{ij} – metric correction, b_{ij} – Osborn anti-symmetric tensor (Osborn 1991)

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  - **redundancy analysis** (Behr & Konechny 2013) $\Rightarrow$ even richer geometry of RG flows (cf. second part of the talk)

Collection of other important results

- $d = 2$:
  - scale invariance implies conformal invariance (Osborn 1991)
  - renormalization of non-linear σ models (Osborn 1991)
  - **g-theorem** for boundary theories (Affleck & Ludwig 1991, Friedan & Konechny 2004)
- $d = 3$
  - form of the local RG anomaly, constraints on possible form of the β-functions (Nakayama 2013)
- $d = 4$:
  - analogue of the c-theorem aka a**-theorem** (Jack & Osborn 1990, Cardy 1990, Komargodski & Schwimmer 2011)
- $d = 6$
  - general form of the local Weyl anomaly (Grinstein & Stergiou & Stone 2013)
- various dimensions: **holographic RG** (Akhmedov 1998, Henningson & Skenderis 1998, Alvarez & Gomez 1999,...)

Part II: Redundancy in 2d QFTs

"Warm up": Simple scalar QFT illustration

- **"redundant coupling"** $\Leftrightarrow$ change in action under variation of this coupling vanishes due to equations of motion (Weinberg 1995)
- local operator that couples to such a coupling = total derivative plus terms $\propto$ e.o.m.'s (which are **pure contact terms**)
- elementary example: (ϕ – scalar; m, Z – couplings)

$$S = \int d^2x \, \frac{1}{2} Z \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\right)$$

- Z couples to local operator $\phi_Z(x)$:

$$\begin{aligned} \phi_Z(x) &\equiv \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi(x) + m^2 \phi^2(x)\right) \\ &= \frac{1}{2}\partial_\mu \left(\phi \partial^\mu \phi\right)(x) + \frac{1}{2}\left[m^2\phi - \partial_\mu \partial^\mu \phi\right](x) \\ &= \frac{1}{2}\partial_\mu \left(\phi \partial^\mu \phi\right)(x) + \frac{1}{2} Z^{-1} \phi \frac{\delta S}{\delta \phi} \end{aligned}$$

General analysis

Action principle and variational calculus

- (renormalized) 2d Euclidean QFT with conserved $T_{\mu\nu}(x)$, such that a **change of scale** is computed via integrating an insertion of $\Theta \equiv g^{\mu\nu} T_{\mu\nu}$:

$$\mu \frac{\partial}{\partial \mu} \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_c = \int d^2y \, \langle \Theta(y) \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_c$$

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- **action principle** (Schwinger 1951): changing λ^i amounts to

$$\frac{\partial}{\partial \lambda^i} \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_c = \int d^2y \, \langle \phi_i(y) \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_c$$

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- **technical trick**: **variational calculus of sources**

$Z(\mu, \lambda^i) \to Z[\mu(x), \lambda^i(x)]$, $g_{\mu\nu} \to g_{\mu\nu}(x) = \mu^2(x) \delta_{\mu\nu}$ such that

$$\mu(x) \frac{\delta \ln Z}{\delta \mu(x)} = \langle \Theta(x) \rangle_c \,, \qquad \mu \frac{\partial}{\partial \mu} \langle \ldots \rangle_c = \int d^2x \, \mu(x) \frac{\delta}{\delta \mu(x)} \langle\langle \ldots \rangle\rangle \ln Z$$

$$\frac{\delta \ln Z}{\delta \lambda^i(x)} = \langle \phi_i(x) \rangle_c \,, \qquad \frac{\partial}{\partial \lambda^i} \langle \ldots \rangle_c = \int d^2x \, \frac{\delta}{\delta \lambda^i(x)} \langle\langle \ldots \rangle\rangle \ln Z \,.$$

Local renormalization anomaly

- effect of a **local scale transformation**: $\Theta(x) = \beta^i \phi_i(x)$ **as operator equation**, i.e. up to **contact terms**

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- possible **contact terms**: encoded ("stored") in expansion constructed via **locality** and 2d covariance:

$$\Theta(x) - \beta^i \phi_i(x) = \frac{\mu^2(x)}{2} R_{(2)} C(\lambda) + \partial_\mu \lambda^i J_i^\mu(x) + \partial^\mu \left[W_i(\lambda) \partial_\mu \lambda^i(x) \right] + \frac{1}{2} \partial^\mu \lambda^i \partial_\mu \lambda^j G_{ij}(\lambda)$$

with:

- $R_{(2)}$ $(\mu(x)^2 R_{(2)} = -\partial_\mu \partial^\mu \ln \mu(x))$ – 2d curvature density
- $J_i^\mu(x)$ – vector currents
- C, W_i and G_{ij} – $\lambda^i(x)$-dependent coefficients

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  - $J_i^\mu(x)$ – vector currents
  - C, W_i and G_{ij} – $\lambda^i(x)$-dependent coefficients
- restricted by
  - **locality** $\Leftrightarrow$ C, W_i and G_{ij} depend on x only through $\lambda^i(x)$
  - type of QFT setup:
    - "strict" power counting: C, W_i and G_{ij} are **functions** of λ^i couplings
    - "loose" power counting: coefficients can have operator content
    - general case: anomaly can have higher derivative terms etc.

Callan–Symanzik equations

$$\left(\mu\frac{\partial}{\partial\mu} - \mathcal{L}_{\hat{\beta}}\right)\langle\phi_{i_1}(x_1)\ldots\Theta(y_1)\ldots\rangle_c = -\langle\partial_\mu J^\mu_{i_1}(x_1)\phi_{i_2}(x_2)\ldots\Theta(y_1)\ldots\rangle_c - \langle\phi_{i_1}(x_1)\partial_\mu J^\mu_{i_2}(x_2)\ldots\Theta(y_1)\ldots\rangle_c + \ldots$$

with: $\mathcal{L}_{\hat{\beta}}$ – Lie-derivative along $\hat{\beta} \equiv \beta^i \frac{\partial}{\partial\lambda^i}$

$\Rightarrow$ correlators do not transform covariantly under scale transformations due to **total derivative terms**!

- in analogy to Lagrangian description, $\partial_\mu J^\mu_i$ are called **redundant operators**

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Question: $\exists$ possibility to quotient out redundant operators?

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Question: $\exists$ possibility to quotient out redundant operators?

- Ansatz:
  - introduce **basis of currents** J^μ_a
  - for a complete basis of scalars $\{\phi_i\}$, we must have

$$\partial_\mu J^\mu_a(x) = {r_a}^i(\lambda)\phi_i(x)$$

as an **operator equation**, i.e. up to contact terms …

Redundancy anomaly

- **Ansatz**: introduce **sources** $\lambda^a_\mu(x)$ (of dimension 1), "store" contact terms in expansion:

$$\partial_\mu J^\mu_a(x) - r^i_a(\lambda)\phi_i(x) = -\mathbb{R}_a(x)$$

with ($\lambda^{a\mu} \equiv \mu^2 g^{\mu\nu}\lambda^a_\nu(x)$)

$$\mathbb{R}_a(x) = k_a\mu^2 R_2(x) + \frac{k_{abc}}{2}\lambda^b_\mu\lambda^{c\mu}(x) + \Gamma_{ba}{}^c\lambda^b_\mu J^\mu_c(x) + r_{ai}{}^b\partial_\mu\lambda^i J^\mu_b(x) + k_{aib}\partial_\mu\lambda^i\lambda^{b\mu}(x) + \frac{k_{aij}}{2}\partial_\mu\lambda^i\partial^\mu\lambda^j(x) + \partial_\mu[k_{ai}\partial^\mu\lambda^i(x) + k_{ab}\lambda^{b\mu}(x)]$$

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- **calculus**: after taking all variational derivatives, set $\mu(x) \mapsto \mu = const$, $\lambda^i(x) \mapsto \lambda^i = const$, $\lambda^a_\mu(x) \mapsto 0$
- in "strict" power counting scenario, all coefficients are **functions** of couplings $\lambda^i(x)$

Example: "differentiation along redundant directions"

For $\partial_\mu J^\mu_a(x) = r_a{}^i(\lambda)\phi_i(x)$ and finite separation $|y-z|>0$,

$$r_a{}^i \frac{\partial}{\partial\lambda^j}\langle J^\nu_b(y)\phi_i(z)\rangle_c = \langle J^\nu_b(y)\,\Gamma^j_{ai}\phi_j(z)\rangle_c + \Gamma^c_{ba}\langle J^\nu_c(y)\phi_i(z)\rangle_c \;,$$

with: $\Gamma^j_{ai} \equiv -\partial_i r_a{}^j - r_{ai}{}^c r_c{}^j$

- ⇒ differentiation of $\langle\ldots\rangle_c$ along **redundant directions** $r_a{}^i\partial_i$ results in **field redefinitions** prescribed by the connection coefficients Γ^j_{ai} and Γ^c_{ba}!
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- in this sense, the operators $\partial_\mu J^\mu_a(x)$ are **redundant**
- **Question**: Can we **quotient out** the "redundant directions" in coupling space such that physical quantities (e.g. c-function, β-functions,…) only dependent on the "transverse" directions?

Modified local renormalization anomaly

$$\Theta(x) - \beta^i(\lambda)\phi_i(x) = \mathbb{D}(x)$$

$$\begin{aligned}\mathbb{D}(x) = & \frac{\mu^2}{2} R_2(x) C + \partial_\mu \lambda^i v_i{}^a(\lambda) J_a^\mu(x) + \lambda_\mu^a \gamma_a{}^b(\lambda) J_b^\mu(x) \\ & + \partial_\mu [W_i \partial^\mu \lambda^i(x) + w_a \lambda^{a\mu}(x)] \\ & + \frac{1}{2} G_{ij} \partial^\mu \lambda^i \partial_\mu \lambda^j(x) + g_{aj} \lambda^{a\mu} \partial_\mu \lambda^j(x) + g_{ab} \frac{1}{2} \lambda^{a\mu} \lambda_\mu^b(x)\end{aligned}$$

with:

- coefficients $v_i{}^a(\lambda)$ defined via $J_i^\mu(x) = v_i{}^a(\lambda) J_a^\mu(x)$
- $\gamma_a{}^b$ – matrix of anomalous dimensions
- all coefficients depend on couplings λ^i in "strict" power counting scenario

Wess–Zumino consistency conditions

$$\left[\mu(x)\frac{\delta}{\delta\mu(x)} - \beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)} - \mathbb{D}(x), \mu(y)\frac{\delta}{\delta\mu(y)} - \beta^j(\lambda)\frac{\delta}{\delta\lambda^j(y)} - \mathbb{D}(y)\right] \stackrel{!}{=} 0$$

$$\left[\mu(x)\frac{\delta}{\delta\mu(x)} - \beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)} - \mathbb{D}(x), \partial_\nu\frac{\delta}{\delta\lambda^b_\nu(y)} - r_b{}^j(\lambda)\frac{\delta}{\delta\lambda^j(y)} + \mathbb{R}_b(y)\right] \stackrel{!}{=} 0$$

$$\left[\partial_\mu\frac{\delta}{\delta\lambda^a_\mu(x)} - r_a{}^i(\lambda)\frac{\delta}{\delta\lambda^i(x)} + \mathbb{R}_a(x), \partial_\nu\frac{\delta}{\delta\lambda^b_\nu(y)} - r_b{}^j(\lambda)\frac{\delta}{\delta\lambda^j(y)} + \mathbb{R}_b(y)\right] \stackrel{!}{=} 0$$

$\Rightarrow$ large number of equations on anomaly coefficients, which **encode geometry** of coupling space!

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- two simple examples:

$$\gamma_a{}^b = -r_a{}^i v_i{}^b + r_{ai}{}^b\beta^i$$

$$\left[\hat{\beta}, \hat{\mathcal{R}}_a\right] = -\beta^i r_{ai}{}^b\hat{\mathcal{R}}_b$$

with:

- $\hat{\beta} \equiv \beta^i\frac{\partial}{\partial\lambda^i} - \beta$ - function vector field
- $\hat{\mathcal{R}}_a \equiv r_a{}^i\frac{\partial}{\partial\lambda^i}$ – redundancy vector fields

Conformal perturbation theory

Conformal perturbation theory setup

- S_0 – 2d Euclidean **CFT** (unitary, discrete spectrum of conformal dimensions) with $\mathcal{G} \times \overline{\mathcal{G}}$ **chiral symmetry algebra**

$$\bar{\partial} J_a = \partial \bar{J}_{\bar{a}} = 0$$

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- **operator product expansions**: ($r.p.$ – regular part)

$$J_a(x) j_b(y) = \frac{\kappa_{ab}}{(x-y)^2} + \frac{i f_{ab}{}^c J_c(y)}{(x-y)} + r.p.$$

$$J_a(x)\Phi_I(y) = \frac{B_{aI}{}^{\bar{b}} \bar{J}_{\bar{b}}(\bar{y})}{(x-y)^2} + \frac{i A_{aI}{}^J \Phi_J(y)}{(x-y)} + r.p.$$

$$\Phi_I(x)\Phi_J(y) = \frac{\delta_{IJ}}{|x-y|^4} + \frac{i A_{IJ}{}^a J_a(y)}{(x-y)(\bar{x}-\bar{y})^2} + \frac{i \bar{A}_{IJ}{}^{\bar{a}} \bar{J}_{\bar{a}}}{(x-y)^2(\bar{x}-\bar{y})}$$

$$+ \frac{C_{IJ}{}^K \Phi_K(y)}{|x-y|^2} + \text{other terms} + r.p.$$

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$$J_a(x) j_b(y) = \frac{\kappa_{ab}}{(x-y)^2} + \frac{i f_{ab}{}^c J_c(y)}{(x-y)} + r.p.$$

$$J_a(x) \Phi_I(y) = \frac{B_{aI}{}^{\bar{b}} \bar{J}_{\bar{b}}(\bar{y})}{(x-y)^2} + \frac{i A_{aI}{}^J \Phi_J(y)}{(x-y)} + r.p.$$

$$\Phi_I(x) \Phi_J(y) = \frac{\delta_{IJ}}{|x-y|^4} + \frac{i A_{IJ}{}^a J_a(y)}{(x-y)(\bar{x}-\bar{y})^2} + \frac{i \bar{A}_{IJ}{}^{\bar{a}} \bar{J}_{\bar{a}}}{(x-y)^2(\bar{x}-\bar{y})} + \frac{C_{IJ}{}^K \Phi_K(y)}{|x-y|^2} + \text{other terms} + r.p.$$

- **perturbation**: $\delta S = \int d^2x\, \lambda^i \phi_i(x)$, with $\{\phi_i\} \subseteq \{\Phi_I\}$

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- **Ward identities**:

$$Q_a(\langle\phi_{i_1}(x_1)\ldots\rangle_c) = 0 = \bar{Q}_{\bar{a}}(\langle\phi_{i_1}(x_1)\ldots\rangle_c)$$

Computation of redundancy data

- **Ansatz**: expand the equation

$$\langle(\bar{\partial}J_a + \partial\bar{J}_{\bar{a}})(x)\Phi_I(y)\rangle_c = ({r_a}^J(\lambda) + {\bar{r}_{\bar{a}}}{}^J(\lambda))\langle\Phi_J(x)\Phi_I(y)\rangle_c$$

in orders of λ's (with: ${r_a}^J(\lambda) = \sum_{\ell>0} \lambda^{i_1} \dots \lambda^{i_\ell}\, {r_{ai_1\dots i_\ell}}^J$)

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- **Results**:

$$\begin{aligned}{r_a}^I(\lambda) &= i\pi\lambda^i {A_{ai}}^I - i\pi^2 B^{\bar{a}}_{ai}{\bar{A}_{\bar{a}j}}{}^I\lambda^i\lambda^j + \mathcal{O}(\lambda^3)\,,\\ {\bar{r}_{\bar{a}}}{}^I(\lambda) &= i\pi\lambda^i{\bar{A}_{\bar{a}i}}{}^I - i\pi^2\bar{B}^a_{\bar{a}i}{A_{aj}}^I\lambda^i\lambda^j + \mathcal{O}(\lambda^3)\,.\end{aligned}$$

where the OPE coefficients were defined via

$$\begin{aligned}J_a(x)\Phi_I(y) &= \frac{{B_{aI}}^{\bar{b}}\bar{J}_{\bar{b}}(\bar{y})}{(x-y)^2} + \frac{i{A_{aI}}^J\Phi_J(y)}{(x-y)} + r.p.\\ \bar{J}_{\bar{a}}(\bar{x})\Phi_I(y) &= \frac{{\bar{B}_{\bar{a}I}}{}^b J_b(y)}{(\bar{x}-\bar{y})^2} + \frac{i{\bar{A}_{\bar{a}I}}{}^J\Phi_J(y)}{(\bar{x}-\bar{y})} + r.p.\end{aligned}$$

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- **Question**: $\exists$? possibility to find basis of currents $\{K_\alpha = {\kappa_\alpha}^a(\lambda)J_a + {\kappa_\alpha}^{\bar{a}}(\lambda)\bar{J}_{\bar{a}}\}$ s.th. action of Q_α "closes" on $\{\phi_i\}$?

The "embedding type" Ansatz

- observation: charges Q_a and $Q_{\bar{a}}$ form a **representation of the chiral algebra** $\mathcal{G} \times \overline{\mathcal{G}}$ on the space of scalars $\{\Phi_I\} = \{\phi_i\} \cup \{\chi_{\tilde{j}}\}$

$$Q_a(\Phi_I(x)) = iA_{aI}{}^J \Phi_J(x) \equiv (\omega_a)_I{}^J \Phi_J(x) \ , \quad [\omega_a, \omega_b]_I{}^J = if_{ab}{}^c (\omega_c)_I{}^J$$

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- **Ansatz**: define $K_\alpha \equiv \kappa_\alpha{}^a J_a + \kappa_\alpha{}^{\bar{a}} \bar{J}_{\bar{a}}$ with charges

$$Q_\alpha(\Phi_I(x)) = \left(\kappa_\alpha{}^a (\omega_a)_I{}^J + \kappa_\alpha{}^{\bar{a}} (\omega_{\bar{a}})_I{}^J\right) \Phi_J(x) \equiv (\rho_\alpha)_I{}^J \Phi_J(x)$$

via the requirements $(\rho_\alpha)_i{}^{\tilde{j}} \stackrel{!}{=} 0$ and $(\rho_\alpha)_{\tilde{i}}{}^j \stackrel{!}{=} 0$

$\Leftrightarrow$ matrices (ρ_α) should form a **reducible representation** of a subalgebra $\mathcal{H} \subset \mathcal{G} \times \overline{\mathcal{G}}$ of the chiral algebra:

$$\rho : \mathcal{H} \to GL(V^{\parallel}) \oplus GL(V^{\perp}) : K_\alpha \mapsto (\rho_\alpha)_I{}^J = \begin{pmatrix} (\rho_\alpha)_i{}^j & 0 \\ 0 & (\rho_\alpha)_{\tilde{i}}{}^{\tilde{j}} \end{pmatrix}$$

The "embedding type" Ansatz – results

- with $K_\alpha \equiv \kappa_\alpha{}^a J_a + \kappa_\alpha{}^{\bar{a}} \bar{J}_{\bar{a}}$ and $Q_\alpha(\Phi_I) = (\rho_\alpha)_I{}^J \Phi_J$ as before, we obtain for the **divergence** $\partial_\mu K^\mu_\alpha = r_\alpha{}^i(\lambda)\phi_i$:

$$r_\alpha{}^i(\lambda) = \pi(\rho_\alpha)_r{}^i \lambda^r + \pi\, \eta_{\alpha r}{}^\beta (\rho_\beta)_s{}^i \lambda^r \lambda^s + \mathcal{O}(\lambda^3)$$

where $\eta_{\alpha r}{}^\beta = \pi(\kappa_\alpha{}^a \kappa_\gamma{}^{\bar{b}} B_{ar\bar{b}} + \kappa_\alpha{}^{\bar{a}} \kappa_\gamma{}^b \bar{B}_{\bar{a}rb})\delta^{\gamma\beta}$

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- the vector field $\hat{\mathcal{R}}_\alpha \equiv r_\alpha{}^i(\lambda)\frac{\partial}{\partial\lambda}$ fulfills the equation

$$[\hat{\beta}, \hat{\mathcal{R}}_\alpha] = -\beta^i \eta_{\alpha i}{}^\beta \hat{\mathcal{R}}_\beta + \mathcal{O}(\lambda^4) \qquad \left(\hat{\beta} \equiv \beta^i \frac{\partial}{\partial\lambda^i}\right)$$

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- the " embedding" Ansatz fulfills all necessary requirements to obtain **redundanty vector fields** $\hat{R}_\alpha$, but in all generality we could allow the more generic Ansatz

$$K_\alpha \equiv \kappa_\alpha{}^a(\lambda) J_a + \kappa_\alpha{}^{\bar{a}}(\lambda) \bar{J}_{\bar{a}}$$

Explicit examples

"All or nothing" examples

- **no perturbation=CFT**: conservation equations $\bar{\partial} J_a = 0$ and $\partial \bar{J}_{\bar{a}} = 0$ hold only up to **contact terms**, which may be derived via the renormalization and redundancy equations
  - results: all non-zero coefficients expressible via **OPE coefficients**

$$
\begin{aligned}
&{r_{ai}}^{\bar{b}} = \pi {B_{ai}}^{\bar{b}} \qquad && {r_{\bar{a}i}}^{b} = \pi {\bar{B}_{\bar{a}i}}^{b} \qquad && {r_{ai}}^{b} = {r_{\bar{a}i}}^{\bar{b}} = 0 \\
&\partial_i {r_a}^j = i\pi {A_{ai}}^j && \partial_i {r_{\bar{a}}}^j = -i\pi {\bar{A}_{\bar{a}i}}^j && \\
&{\Gamma_{ab}}^c = i\pi {f_{ab}}^c && {\Gamma_{\bar{a}\bar{b}}}^{\bar{c}} = i\pi {f_{\bar{a}\bar{b}}}^{\bar{c}} &&
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$$\partial_i r_a{}^j = i\pi A_{ai}{}^j \qquad \partial_i r_{\bar{a}}{}^j = -i\pi \bar{A}_{\bar{a}i}{}^j$$

$$\Gamma_{ab}{}^c = i\pi f_{ab}{}^c \qquad \Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} = i\pi f_{\bar{a}\bar{b}}{}^{\bar{c}}$$

- **perturbation by all scalars** Φ_I: $\delta S = \int d^2x \, \lambda^I \Phi_I(x)$
  - possible choice of basis for currents K_α: $K_\alpha = \delta_\alpha{}^a J_a + \delta_\alpha{}^{\bar{a}} \bar{J}_{\bar{a}}$, i.e. the matrices ρ_α simply form a representation of the full chiral algebra $\mathcal{G} \times \overline{\mathcal{G}}$ on the space of operators Φ^I!
  - the redundancy coefficients $r_\alpha{}^I$ ($\hat{R}_\alpha = r_\alpha{}^I \frac{\partial}{\partial \lambda^I}$) are given in terms of the **connection coefficients** of $\mathcal{G} \times \overline{\mathcal{G}}$:

$$r_a{}^I(\lambda) = -\Gamma_{aR}{}^I \lambda^R + \mathcal{O}(\lambda^3) \, , \Gamma_{aR}{}^I = -\partial_R r_a{}^I - r_{aR}{}^c r_c{}^I$$

(and analogously for $r_{\bar{a}}{}^I(\lambda)$)

The conformal $SO(3)$ model

- $SU(2)_k$ WZW model, perturbed by **current-current operators**

$$\delta S = \int d^2x \, \lambda^i \phi_i(x) \,, \qquad \phi_i = \frac{1}{k} J_i \bar{J}_3 \qquad i \in \{1, 2, 3\}$$

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- result of redundancy analysis:
  - perturbed theory has two conserved currents

$$J_L = \lambda^1 J_1 + \lambda^2 J_2 + \lambda^3 J_3\,, \qquad J_R = \bar{J}_3\,. \tag{1}$$

  - redundancy vector fields are the **rotation vector fields** in the 3d space of couplings

$$\widehat{R}_a^{(0)} = i\pi \varepsilon_{ai}{}^{j} \lambda^i \partial_j + \mathcal{O}(\lambda^3)\,. \tag{2}$$

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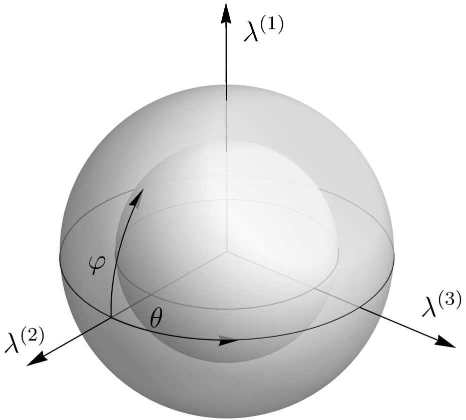
- $\mathbf{k = 1}$: $SU(2)_1$ WZW model $\hat{=}$ **free boson at self-dual radius**
  - $\lambda^1 = \lambda^2 = 0$, $\lambda^3 = \frac{R - \frac{1}{R}}{R + \frac{1}{R}} \Rightarrow J_3 \bar{J}_{\bar{3}} \hat{=}$ boson radius changing operator
  - **T-duality**: $R \mapsto \frac{1}{R} \Rightarrow \lambda^3 \mapsto -\lambda^3$

The conformal $SO(3)$ model



Orbits of the action of the redundancy vector fields $\widehat{R}_a$.
T-duality: $\lambda^3 \mapsto -\lambda^3$ via **continuous** rotation!

The "sausage" model

Fateev, Onofri & Zamolodchikov 1992: S_0 – $SU(2)_k$ WZW model, perturbed by **current-current operators**

$$\phi_{(13)} \equiv \frac{1}{\sqrt{2}}\left(: J_1\bar{J}_{\bar{3}} : + : J_3\bar{J}_{\bar{1}} :\right)$$
$$\phi_{(22)} \equiv : J_2\bar{J}_{\bar{2}} :$$
$$\phi_{\widetilde{(13)}} \equiv \frac{1}{\sqrt{2}}\left(: J_1\bar{J}_{\bar{1}} : - : J_3\bar{J}_{\bar{3}} :\right)$$

- result of redundancy analysis: we find the Abelian redundancy vector field

$$\hat{\mathcal{R}} \equiv 2\pi i\left(\lambda^{\widetilde{(13)}}\frac{\partial}{\partial\lambda^{(13)}} - \lambda^{(13)}\frac{\partial}{\partial\lambda^{\widetilde{(13)}}}\right) + \mathcal{O}(\lambda^3)$$

  which implements **rotations in the** $\lambda^{(13)}$–$\lambda^{(\widetilde{13})}$ plane in coupling space
- the β- function coefficients may be expressed in the basis $r = \sqrt{(\lambda^{(13)})^2 + (\lambda^{(\widetilde{13})})^2}$, $\varphi = \arctan(\lambda^{(\widetilde{13})}/\lambda^{(13)})$ and $z = \lambda^{(22)}$, with

$$\beta^{\varphi} = 0 + \mathcal{O}(\lambda^4)$$

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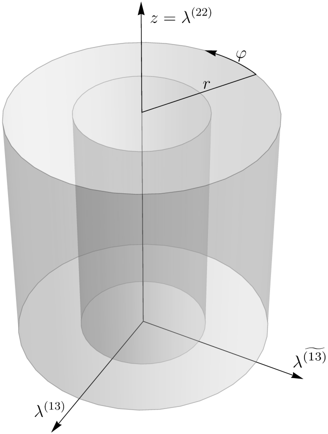

Figure: Orbits under the action of the redundancy vector field $\hat{R} \propto \frac{\partial}{\partial\varphi}$.

Outlook

Summary/Overview of 1310.4185

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- results in three levels of generality:
    - **generic**:
      local renormalization and redundancy **anomaly equations** + **Wess-Zumino consistency conditions**
      $\Rightarrow$ relations on RG coefficients
    - **conformal perturbation theory**:
      explicit formulae for **RG-data in terms of OPE coefficients** up to NLO (beta functions, redundancy vector fields, c-function,...)
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- **geometry of coupling space –** foliations, quotients, polar representations, ...
- higher-dimensional/boundary theory (in particular in 4d)
- holographic RG analysis?

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